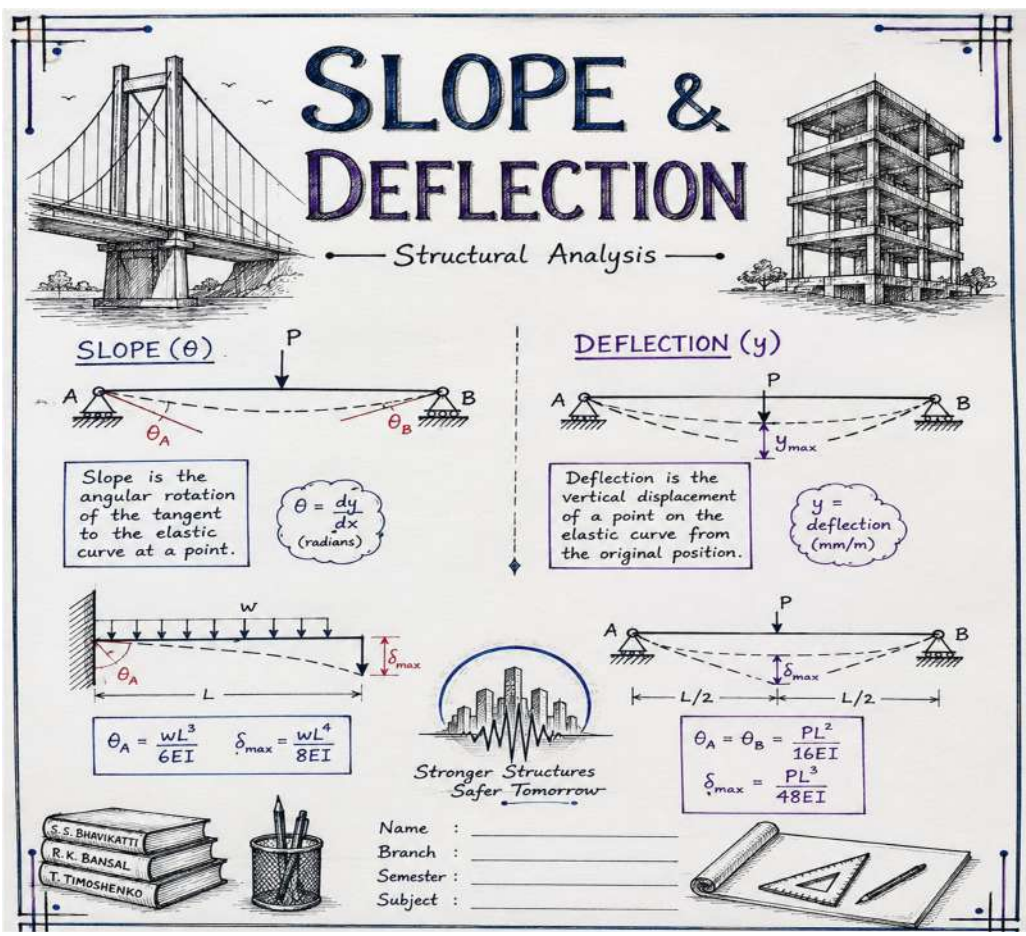
# R.C.Patel College Of Engineering & Polytechnic, Shirpur

## Department of Civil Engineering

**Course Title-** Theory of Structure **Course Code** -315313

**Programme Name** -Civil Engineering **Semester**-Fifth

| Unit | Title | COs | Learning hours | R Level | U Level | A Level | Total Marks |
|---|---|---|---|---|---|---|---|
| I | Slope and Deflection | CO1 | 12 | 2 | 4 | 8 | 14 |



# Unit - 02 Unit-1

01

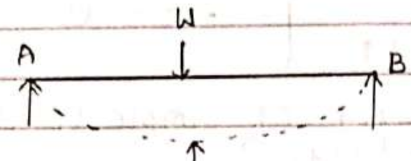
## — slope and Deflection —

Prof. A.H. Patil

* **Concept of slope and Deflection, stiffness of Beams:-**

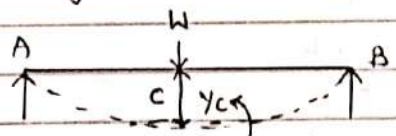
* **Deflection curve :-**

Due to loading on Beam, the span of beam bends showing sagging type curve which is also known as elastic curve



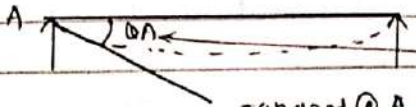

elastic curve ( bending of Beam)

* **Deflection :-** The deflection of beam at any point on beam is the distance between its position before and after loading.




Deflection at pt C

* **slope :-** The slope of beam at any point is the angle made by tangent at that point with original axis of beam.




slope at pt-A

Tangent @ A

* **stiffness :-** The ability of a beam to resist the elastic deformation is called stiffness. OR

The ratio of max. deflection of beam to the span of beam is called as stiffness of beam.

$$K = \frac{Y_{max}}{L}$$ OR

The rate of change of deflection with respective to span length is known as slope.

$$\theta = \frac{d^2y}{dx^2} \text{ in radians.}$$

## * Relation between slope, Deflection and Radius of Curvature :-

We know that, flexural formula,

$$\frac{M}{I} = \frac{\sigma}{y} = \frac{E}{R}$$

$$\therefore \frac{M}{I} = \frac{E}{R}$$

$$\boxed{\frac{1}{R} = \frac{M}{EI}} \quad --- \text{①}$$

But from theory of simple (pure) bending,

$$\frac{1}{R} = \frac{(d^2y/dx^2)}{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}$$

$$\boxed{\frac{1}{R} = \frac{d^2y}{dx^2}} \quad \text{(approximately)} \quad --- \text{②}$$

Equating (1) and (2)

$$\frac{d^2y}{dx^2} = \frac{M}{EI}$$

$$EI\frac{d^2y}{dx^2} = M_x \quad --- \text{③}$$

Equation ③ is governing differential equation for slope and deflection.

where, EI = Flexural rigidity

Integrating equation ③ w.r.to $x$

$$\boxed{EI\frac{dy}{dx} = \int M_x\,dx + C_1} \quad \text{——} \quad \text{④}$$

Equation ④ is basic slope equation

Now integrating eq^n. ④ w.r.to $x$.

$$\boxed{EI \cdot y = \int (M_x\,dx)\cdot dx + C_1\cdot dx + C_2} \quad \text{—} \quad \text{⑤}$$

Equation ⑤ is basic deflection equation

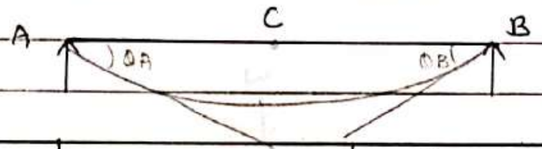
where $C_1$ & $C_2$ = constants of integration, which can be calculated boundary conditions of beam.

## * Boundary conditions of Beam :-

1) fixed Beam :-

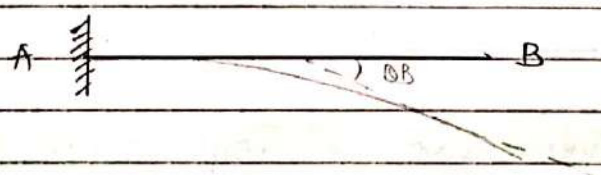

| | | | |
|---|---|---|---|
| $\theta A = 0$ | $\theta c = 0$ | $\theta B = 0$ | slopes |
| $y_A = 0$ | $y_C = max$ | $y_B = 0$ | Deflections |

2) simply supported Beam :-




| | | | |
|---|---|---|---|
| $\theta A \neq 0$ | $\theta c = 0$ | $\theta B = max$ | slopes |
| $y_A = 0$ | $y_c = max$ | $y_B = 0$ | Deflection |

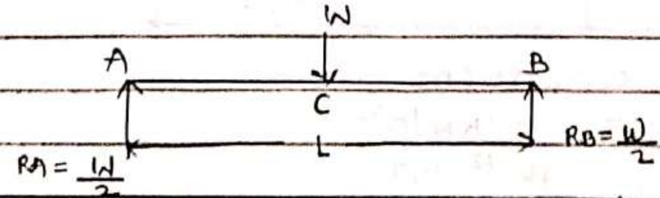
3) cantilever Beam :-




| | | |
|---|---|---|
| $\theta A = 0$ | $\theta B = max$ | slope |
| $y_A = 0$ | $y_B = max$ | Deflection |

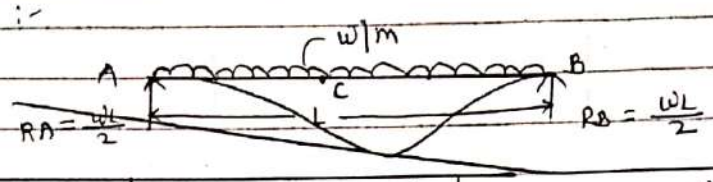
## * Standard formulae for slope and Deflection for Different Beams :-

case 1) simply supported Beam (SSB) carrying a central point load :-



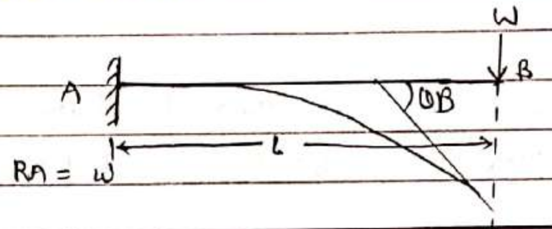

| | | |
|---|---|---|
| $\theta A = \frac{WL^2}{16EI}$ rad | $y_c = \frac{WL^3}{48EI}$ mm | $\theta B = \frac{WL^2}{16EI}$ rad |

**Case-2 Simply supported Beam (SSB) carrying entire UDL :-**



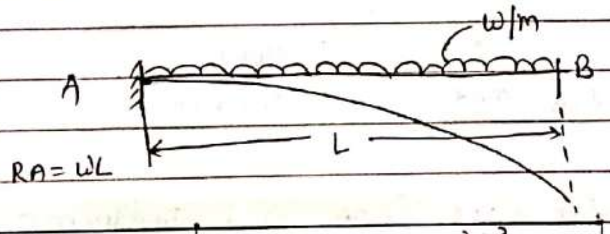

| | | |
|---|---|---|
| $\theta A = \frac{WL^3}{24EI}$ | $y_C = \frac{5}{384}\frac{WL^4}{EI}$ | $\theta B = \frac{WL^3}{24EI}$ |

**Case-3 Cantilever Beam carrying point load at free end.**




| | |
|---|---|
| $\theta A = 0$<br>$y_A = 0$ | $\theta B = \frac{WL^2}{2EI}$<br>$y_B = \frac{WL^3}{3EI}$ |

**Case-4 Cantilever Beam carrying entire UDL :-**




| | |
|---|---|
| $\theta A = 0$<br>$y_A = 0$ | $\theta B = \frac{WL^3}{6EI}$<br>$y_B = \frac{WL^4}{8EI}$ |

* Important conversions :-

$1\ N/mm^2 = 10^3\ kN/m^2$

$1\ mm^4 = 10^{-12}\ m^4$

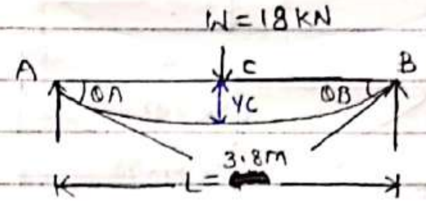
$1\ kN \cdot m^2 = 10^9\ N \cdot mm^2$

$1\ kN \cdot mm^2 = 10^3\ N \cdot mm^2$

slope = $\theta A = \theta B = \frac{WL^2}{16EI}$ rad
deflection = $y_c = \frac{WL^3}{48EI}$ } S.S.B. central point load.

point load

Ex:17) A simply supported beam of span 3.8m carries a central point load of 18 KN. find the maximum slope and deflection of the beam, if $I_{xx} = 2\times10^8 mm^4$ and E = 200 Gpa




Given :- $W = 18 \text{ kN} = 18\times10^3 N$

$L = 3.8m = 3800 mm$

$I_{xx} = 2\times10^8 mm^4$

$E = 200 \text{ Gpa} = 200\times10^3 N/mm^2$

find $\theta max = ?$, $y_{max} = ?$

when SSB carries central point load, maximum slope i.e. $\theta max$ occurs at supports i.e. at A and B. similarly maximum deflection i.e. $y_{max}$ occurs at mid span i.e. at point C.

∴ To find $\theta max$, $\theta A = \theta B$

$$\theta A = \theta B = \frac{WL^2}{16EI} = \frac{18\times10^3 \times 3.8^2}{16\times200\times10^3\times2\times10^8}$$

$$= \frac{18\times10^3\times3800^2}{16\times200\times10^3\times2\times10^8}$$

$$= 0.000406$$

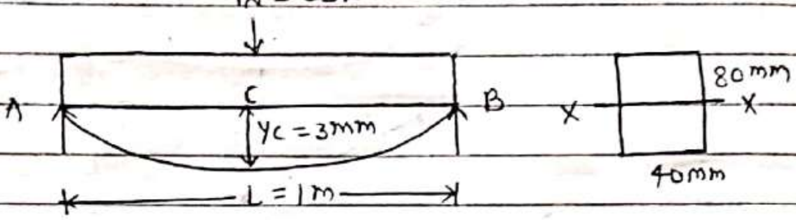
$$\boxed{\theta A = \theta B = 4.06\times10^{-4} \text{ rad.}}$$

To find $y_{max}$ i.e. $y_c$

$$y_c = \frac{WL^3}{48EI} = \frac{18\times10^3\times3800^3}{48\times200\times10^3\times2\times10^8}$$

$$\boxed{y_c = 0.514 mm}$$

Ex:2) A cast iron beam 40mm wide and 80 mm deep is simply supported over a span of 1m. It showed a deflection of 3mm under a central load of 25kN. Calculate the modulus of elasticity of the material and the slope at the supports.




Give Data :-

$b = 40mm$ , $d = 80mm$

$W = load = 25KN = 25 \times 10^3 N$ [converting into N]

length $(L) = 1.0m = 1000mm$ [converting into mm]

Now we have to find $E = ?$ (modulus of Elasticity)

so to find E, by using formula for central deflection of SSB.

$$y_c = \frac{WL^3}{48EI}$$

$$3 = \frac{25 \times 10^3 \times 1000^3}{48 \times E \times I}$$

Here we have to find I

$$I = I_{xx} = \frac{bd^3}{12} =$$ [Here downward central load, so beam deflection x-x axis]

$$= \frac{40 \times 80^3}{12}$$

$$I_{xx} = 1.706 \times 10^6 mm^4$$

$$3 = \frac{25 \times 10^3 \times 1000^3}{48 \times E \times 1.706 \times 10^6}$$

$$E = \frac{25 \times 10^3 \times 1000^3}{3 \times 48 \times 1.706 \times 10^6}$$

$$\boxed{E = 101.77 \times 10^3 N/mm^2}$$

To find $\theta A = \theta B = \frac{WL^2}{16EI} = \frac{25 \times 10^3 \times 1000^2}{16 \times 101.77 \times 10^3 \times 1.706 \times 10^6}$

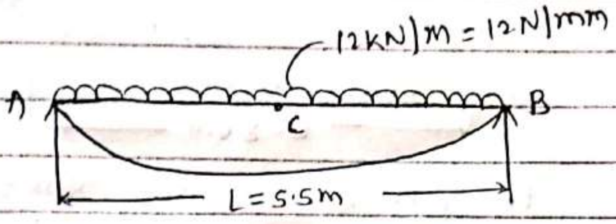
$$\boxed{\theta A = \theta B = 9 \times 10^{-3} rad} = 0.002 rad.$$

* 1kN/m = 1N/mm

$1 kN/m = \frac{1 \times 10^3}{10^3} N/mm$

udl

Ex: 3) A simply supported beam of span 5.5m carries a UDL of 12 kN/m over the entire span. find the slopes at supports and deflection at mid-span if $EI = 4 \times 10^{13} N \cdot mm^2$.




Given Data :- W (Load = 12kN/m = 12 N/mm

span = 5.5m = 5500mm

$EI = 4 \times 10^{13} N \cdot mm^2$

we have to find :- slopes at supports $\theta A = \theta B = ?$

Deflection at mid span $= y_c = ?$

By using standard formula we find slopes,

$$\theta_A = \theta_B = \frac{WL^3}{24 EI}$$

$$= \frac{12 \times 5500^3}{24 \times 4 \times 10^{13}}$$

$$\boxed{\theta A = \theta B = 2.07 \times 10^{-3} rad}$$

To find deflection at mid span i.e. at point 'c'

$$y_c = \frac{5}{384} \frac{WL^4}{EI}$$

$$= \frac{5}{384} \frac{12 \times 5500^4}{4 \times 10^{13}}$$

$$\boxed{y_c = 3.57 mm}$$

Ex: 4) A. S.S.B. carries UDL of 4 kN/m over entire span of 4m. find the deflection of mid span in terms of EI

→ Given :- W = 4 kN/m = 4 N/mm

Find :- y at mid-span $= y_{max} = ?$

$$y_{max} = y_c = \frac{5}{384} \frac{WL^4}{EI} = \frac{5}{384} \times \frac{4 \times 4000^4}{EI}$$

$$\boxed{y_c = \frac{1.33 \times 10^{13}}{EI}}$$

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For Ex:-

$I = 2\times10^{6}\,mm^4 = (2\times10^{6})\times10^{-12}\,m^4 = 2\times10^{-6}\,m^4$

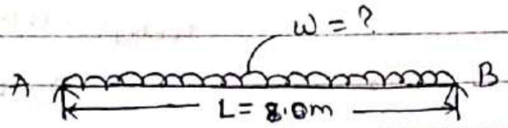
i.e $2\times10^{6}\,mm^4 = 2\times10^{-6}\,m^4$

$I = 50\times10^{-6}\,m^4 = 50\times10^{6}\,mm^4.$

EX:-5) A simply supported beam of 8m carries a UDL if the maximum deflection is not to exceed 20mm calculate the value of UDL.

TAKE $E = 200\,Gpa$ and $I = 50\times10^{-6}\,m^4$

→




Given :- Length of Beam = 8m = 8000 mm

$E = 200\,Gpa = 200\times10^{13}\,N/mm^2$

$I = 50\times10^{-6}\times10^{+12}\,mm^4$ $[\because 1mm^4 = 10^{-12}\,m^4]$

$= 50\times10^{6}\,mm^4$

$y_{max} = y_c = 20\,mm$

We have to find :- Intensity of UDL = ω = ?

∴ so we know that for SSB carrying entire UDL max. deflection is given by -

$$y_{max} = y_c = \frac{5}{384}\frac{\omega L^4}{EI}$$

$$20 = \frac{5}{384}\times\frac{\omega\times8000^4}{200\times10^{13}\times50\times10^{6}}$$

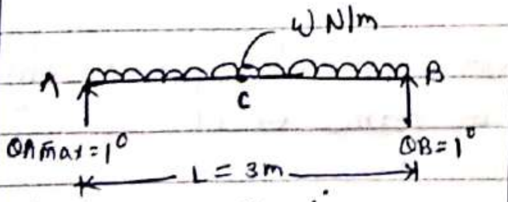
$$20 = 5.33\,\omega$$

$$\boxed{\omega = \frac{20}{5.33}}$$

∴ $\omega = 3.75\,N/mm$ or

$\omega = 3.75\,kN/m$. Ans

EX:6) A beam of span 3m is simply supported and carries udl of ω N/m, if the slope at the end is not to exceed 1°, Find the maximum deflection.




* Find max. deflection $y_{c\,max}$ = ?

for S.S.B. with full UDL

$$\theta_A = \theta_B = \frac{\omega L^3}{24EI}$$

$$\left(1^\circ\times\frac{\pi}{180}\right) = \frac{\omega(3000)^3}{24\times EI}$$

$$0.01745 = \frac{\omega}{EI}\times1.125\times10^{9}$$

$$\therefore \omega = \frac{\left(1^\circ\times\frac{\pi}{180}\right)\times24}{(3000)^3}$$

$$\boxed{\omega = 1.551\times10^{-11}\,EI}$$

Now, for SSB with full udl,

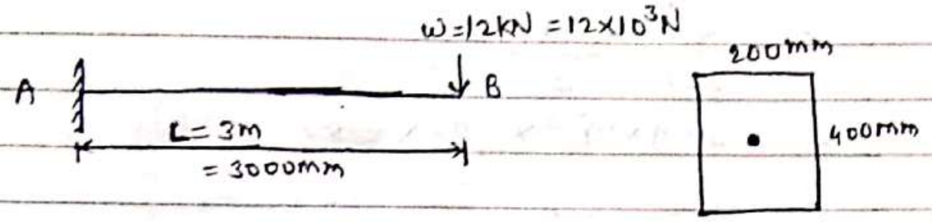
$$y_c = \frac{5}{384}\frac{\omega L^4}{EI}$$

$$y_c = \frac{5}{384}\frac{1.551\times10^{-11}\cancel{EI}\times(3000)^4}{\cancel{EI}}$$

$$\boxed{y_c = 16.34\,mm}$$

* Cantilever Beam

EX07) A wooden cantilever beam of span 3m has a cross-section 200mm wide and 400 mm deep. Find the slope and deflection of the free end when a load of 12kN is applied there. Take $E = 100\ kN/mm^2$




Given data :- span = $L = 3m = 3000mm$ , $w = 12kN = 12\times10^3 N$

width = $b = 200mm$ , depth = $400mm$

$E = 100kN/mm^2 = 100\times10^3 N/mm^2$

find :- $\theta_B = ?$

$y_B = ?$

To find slope at free end i.e at B :-

$$\theta_B = \frac{wL^2}{2EI} = \frac{12\times10^3\times(3000)^2}{2\times100\times10^3\times1.067\times10^9}$$

$$\left(\because I = \frac{bd^3}{12} = \frac{200\times400^3}{12} = 1.067\times10^9\ mm^4\right)$$

$$\therefore \boxed{\theta_B = 5.060\times10^{-4}\ rad} \quad \text{Ans}$$

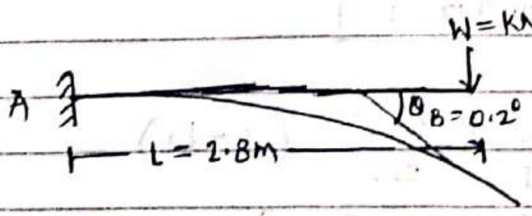
Now to find deflection at free end i.e. at 'C'

$$\therefore y_B = \frac{wL^3}{3EI}$$

$$= \frac{12\times10^3\times(3000)^3}{3\times(100\times10^3\times1.067\times10^9)}$$

$$\boxed{y_B = 1.01mm} \quad \text{Ans}$$

EX:08 For a cantilever of span 2.8m carrying a point load 'W' at the free end, the maximum slope was 0.2 degree.




Given :- Load = W kN , span = 2.8m , $\theta_B = 0.2^\circ$

find :- $y_{max} = y_B = ?$

Now, $y_{max} = y_B = \frac{wL^3}{3EI}$

$\cancel{y_B} = \frac{\cancel{[illegible]}}{\cancel{[illegible]}} \times \frac{\cancel{[illegible]}}{8}$

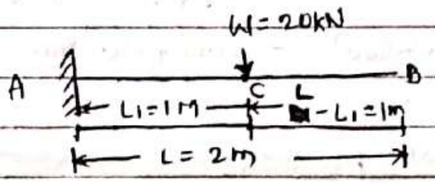
$= \frac{wL^2}{2EI} \times \frac{2L}{3}$

$= 3.49 \times 10^{-3} \times \frac{2}{3} \times \cancel{[illegible]}\ (2800)$

$\boxed{y_B = \cancel{[illegible]}\ mm}$ $\boxed{y_B = 6.514\ mm}$

EX:-09 A cantilever 2m long is 120mm wide and 200mm deep it is subjected to a point load of 20kN at 1m from fixed end. Calculate the slope and deflection at the free end of a cantilever. Take E = 200Gpa.

→




Given data :- $w = 20kN = 20 \times 10^3 N$

$L = 2m = 2000mm$

$b = 120mm$, $d = 200mm$

$E = 200\ Gpa = 200 \times 10^3\ N/mm^2$

find $\theta_B = ?$, $y_B = ?$

Now we find ~~[illegible]~~ maximum slope,

$\theta_{max.} = \theta_B = \theta_C = \frac{WL_1^2}{2EI} = \frac{20 \times 10^3 \times 1000^2}{2 \times 200 \times 10^3 \times 80 \times 10^6}$

$\because I = \frac{bd^3}{12} = \frac{120 \times 200^3}{12} = 80 \times 10^6 mm^4$

$\boxed{\theta_B = \theta_C = 6.25 \times 10^{-4}\ rad}$

$y_B = y_{max} = \frac{WL_1^3}{3EI} + \frac{WL_1^2}{2EI}(L - L_1)$

$= \left[\frac{20 \times 10^3 \times 1000^3}{3 \times 200 \times 10^3 \times 80 \times 10^6}\right] + \left[\frac{20 \times 10^3 \times 1000^2}{2 \times 200 \times 10^3 \times 80 \times 10^6}\right] \times 1000$

$y_B = 0.416 + 0.625$

$\boxed{y_B = 1.041\ mm}$

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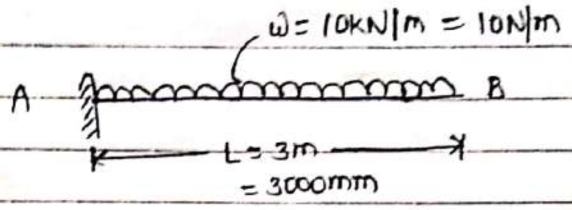
EX:-10) A cantilever beam is carrying udl of 10KN/m over full length of 3m. calculate slope and deflection of the free end. consider young's modulus as $2.1 \times 10^5$ mpa and moment of inertia as $180 \times 10^6$ mm$^4$

→




Given Data :- $E = 2.1 \times 10^5$ mpa

$= 2.1 \times 10^5$ N/mm$^2$

Load = $\omega$ = 10 KN/m = 10N/m

length = $L$ = 3m = 3000mm

$I = 180 \times 10^6$ mm$^4$

Now find :- slope at free end $= \theta_B = ?$

$$\therefore \theta_B = \frac{\omega L^3}{6EI} = \frac{10 \times 3000^3}{6 \times 2.1 \times 10^5 \times 180 \times 10^6}$$

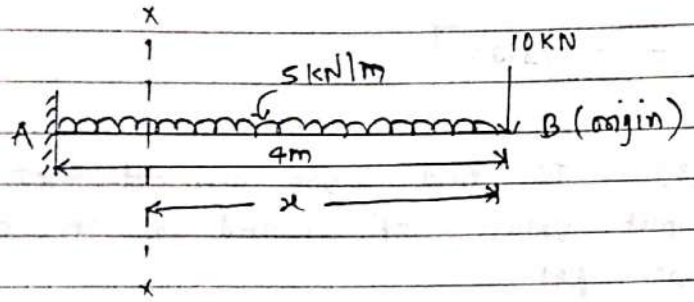
$$\boxed{\theta_B = 1.190 \times 10^{-3} \text{ rad}}$$

For cantilever beam subjected to full udl, deflection at free end is given by,

$$y_B = \frac{\omega L^4}{8EI} = \frac{10 \times 3000^4}{8 \times 2.1 \times 10^5 \times 180 \times 10^6}$$

$$\boxed{y_B = 2.678 \text{mm}}$$

A cantilever beam of 4m length carrying udl 5 KN/m over entire length and a point load of 10 kN at free end. find the slope and deflection at free end.




step 1) Taking section x-x at a dist. x from origin the governing Differential eqn.

$$EI \frac{d^2y}{dx^2} = Mx$$

$$EI \frac{d^2y}{dx^2} = \left| -10x \right| - \left| 5 \times x \times \frac{x}{2} \right|$$

$$\therefore EI \frac{d^2y}{dx^2} = -10x - 2.5x^2 \quad \text{---i)}$$

Integrating w.r.t. x we get

$$EI \frac{dy}{dx} = -\frac{\cancel{10}^{5} x^2}{\cancel{2}_{1}} - \frac{2.5x^3}{3} + c_1 \quad \text{--- ii)}$$

Again integrating w.r.t. x we get

$$EIy = \cancel{-10x} - \frac{5x^3}{3} - \frac{\cancel{1.67}}{0.83}\frac{x^4}{4} + c_1 x + c_2 \quad \text{--- iii)}$$

step 2) To find $c_1$ & $c_2$

condition 1) At A $x = 4m$, $\frac{dy}{dx} = 0$

put in eqn (ii)

$$EI \times 0 = -5 \times 4^2 - \frac{2.5 \times 4^3}{3} + c_1$$

$$= -80 - 53.33 + c_1$$

$$= -133.33$$

$$\therefore \boxed{c_1 = 133.33}$$

condition 2:- At A, $x = 4m$ $y = 0$

put in eqn (iii)

$$0 = \frac{-5 \times 4^3}{3} - \frac{\cancel{1.67}}{0.83} \times \frac{4^4}{4} + 133.33 \times 4 + c_2$$

$$0 = -106.67 - 53.12 + \cancel{133.33}^{533.32} + C_2$$

$$0 = C_2 + 373.32$$

$$\boxed{C_2 = -373.32}$$

step 3) To find slope and deflection at free end put values of $C_1$ and $C_2$ in eq$^n$ (ii) & (iii) we get

$$EI \frac{dy}{dx} = \frac{-5x^2}{2} - \frac{2.5x^3}{3} + 133.33 \quad \text{— slope eq}^n \text{ iv)}$$

$$EIy = \frac{-5x^3}{3} - \frac{2.5x^4}{3} + 133.33x - 373.32 \quad \text{— defl}^n \text{ eq}^n \text{ v)}$$

To find slope at 'B'

At 'B' $x = 0$

put $x = 0$ in eq$^n$ (iv)

$$\left(EI \frac{dy}{dx}\right)_B = 0 - 0 + 133.33$$

$$EI\theta_B = +133.33$$

$$\boxed{\theta_B = \frac{133.33}{EI}} \quad \text{slope at 'B'}$$

To find deflection at 'B'

At 'B' $x = 0$

put $x = 0$ in deflection eq$^n$. we get

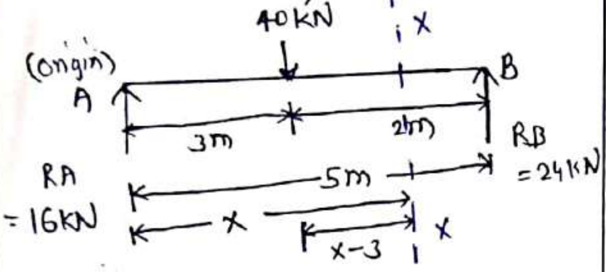
$$EIy = 0 - 0 + 0 - 373.32$$

$$EIy = -373.32$$

$$\boxed{y_B = \frac{-373.32}{EI}} \quad \text{Deflection at 'B'}$$

## 1 * Numericals on Macaulay's Method to find slope and Deflection of simply supported Beam.

Ex:-1) A simply supported beam of span 5m. carries a point load of 40 KN at 3m. from the left support. calculate the slope and deflection under the point load in terms of EI use macaulay's method.




firstly we calculate support Reaction i.e. RA and RB.

$\Sigma F_y = 0$ $[\overset{+}{\uparrow} \downarrow]$

$\therefore$ $RA - 40 + RB = 0$

$\boxed{RA + RB = 40\ KN}$ --- i)

Taking moment @ A

$RA \times 0 + 40 \times 3 - RB \times 5 = 0$

$RB = \frac{120}{5} = 24\ KN$

put $RB = 24\ KN$ in eqn. (1)

(1) → $RA + 24 = 40$

$\boxed{RA = 16 KN}$

* Now consider a section X-X at a distance 'x' from support A. (+)(-)

$M_x = RA \times (x) - 40 \times (x-3)$

but $M_x = EI \frac{d^2y}{dx^2}$

$\boxed{EI \frac{d^2y}{dx^2} = RA \cdot x - 40(x-3)}$ - (1)

Integrating Equation (1), we get

$\boxed{EI \frac{dy}{dx} = RA \cdot \frac{x^2}{2} - \frac{40(x-3)^2}{2} + C_1}$

--- (2) slope equation

Again integrating equation (2) we get.

$\boxed{EIy = RA \frac{x^3}{6} - \frac{40(x-3)^3}{6} + C_1 x + C_2}$

---- (3) Deflection equation.

To calculate $C_2$ apply boundary condition put $x=0$, $y=0$ in equation (3) we get.

$EIy = RA \frac{x^3}{6} + C_1 x + C_2$

$0 = 0 + 0 + C_2$

$\boxed{C_2 = 0}$

* To calculate $C_1$ put $x = 5m$ and $y = 0$ in Equation (3) we get

$EIy = \frac{RA \times (5)^3}{6} - \frac{40(5-3)^3}{6} + C_1 \times 5 + 0$

$0 = \frac{16 \times (5)^3}{6} - \frac{40(5-3)^3}{6} + C_1 \times 5 + 0$

$0 = 333.33 - 53.33 + C_1 \times 5$

$-C_1 \times 5 = 333.33 - 53.33$

$-C_1 = 56$ $\boxed{C_1 = -56}$

* Now To calculate slope under Point load put $x = 3m$ and $\frac{dy}{dx} = \theta_c$ in Equation (2).

$$EI \frac{dy}{dx} = R_A \frac{x^2}{2} - \frac{40(x-3)^2}{2} + C_1$$

$$EI\,\theta_C = \frac{16 \times (3)^2}{2} - \frac{40(3-3)^2}{2} + (-56)$$

$$= 72 - 0 - 56$$

$$EI\,\theta_C = 16$$

$$\boxed{\theta_C = \frac{16}{EI}}$$ Ans

To calculate deflection under point load put $x = 3m$, $y = y_{max}$ in equation (3)

$$EIy = R_A \frac{x^3}{6} - \frac{40(x-3)^3}{6} + C_1 x + C_2$$

$$EIy_{max} = \frac{16 \times 3^3}{6} - \frac{40(3-3)^3}{6} + (-56 \times 3) + 0$$

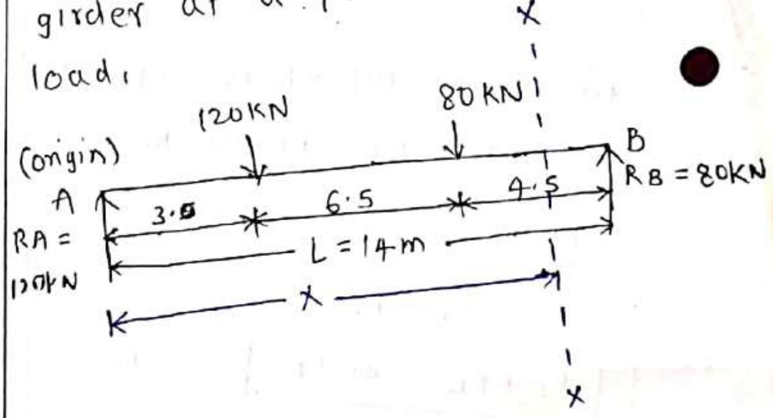
$$= 72 - 0 - 56 \times 3 + 0$$

$$EIy_{max} = -96$$

$$\boxed{y_{max} = \frac{-96}{EI}}$$ Ans.

Ex:- 02 ( S.S.B → Two Point load )

A Horizontal girder of steel having uniform section is 14m long and simply supported at its ends. It carries concentrated loads of 120kN and 80 kN at two points 3m and 4.5m from the two ends respectively. moment of inertia for the section of the girder is $16 \times 10^8 mm^4$ and E for steel is $2.1 \times 10^5$ MPa. calculate the deflection of the girder at a point under 80 kN load.




→ Given Data :- $L = 14m$, $w_1 = 120kN$, $w_2 = 80kN$

$I = 16 \times 10^8 mm^4$

$= (16 \times 10^8) \times 10^{-12} m^4$

$\boxed{I = 16 \times 10^{-4} m^4}$

$E = 2.1 \times 10^5 MPa = 2.1 \times 10^5 N/mm^2$

$= (2.1 \times 10^5) \times 10^3 = 2.1 \times 10^8 kN/m^2$

* step 1 To find support reactions RA and RB [ $\overset{+}{\uparrow}$ $\overset{-}{\downarrow}$ ]

$\Sigma f_y = R_A - 120 - 80 + R_B$

$\boxed{R_A + R_B = 200}$ --- (1)

Taking moment @ point A'

$\Sigma m_A = 120 \times 3 + 80 \times 9.5 - R_B \times 14$

$0 = 1120 - 14 R_B$

$\therefore R_B = \frac{1120}{14}$ $\boxed{\therefore R_B = 80 KN}$

substitute the value of $R_B$ in equation (1) we get

(1) → $R_A + 80 = 200$

$R_A = 200 - 80$

$\boxed{R_A = 120 KN}$

* step (2) TO construct slope and deflection equation by macaulay's method.

consider support A as origin and take section x-x at a dist. x from A as shown in fig. The general bending moment equation at a dist. 'x' from 'A' is.

$EI \frac{d^2y}{dx^2} = m_x$

$$\boxed{\begin{aligned} = \left| \overset{R_A}{120} \times x \right| - \left| 120 \times (x-3) \right| \\ - \left| 80 \times (x - 9.5) \right| \quad \text{---i)} \end{aligned}}$$

$\because$ [$m_x$ = Bending moment at section x-x and by sign convention + −]

Integrating equation (1) w.r.t. x

we get

$EI \frac{dy}{dx} = \left| 120 \times \frac{x^2}{2} + C_1 \right| - \left| 120 \frac{(x-3)^2}{2} \right| - \left| \frac{80 (x-9.5)^2}{2} \right|$ ... slope eqn (ii)

Integrating eqn. (ii) w.r.t. x we get.

$EIy = \left| \frac{120}{2} \cdot \frac{x^3}{3} + C_1 x + C_2 \right| - \left| \frac{120}{2} \frac{(x-3)^3}{3} \right| - \left| \frac{80}{2} \frac{(x-9.5)^3}{3} \right|$

.... deflection eqn. (iii)

* Note :- $C_1$ and $C_2$ are constants of integration.

step 3 :- Apply boundary conditions to find the values of constant $C_1$ and $C_2$

* condition 1:- At A, x = 0, y = 0

substituting these values in deflection equation (iii) upto first bracket only we get

$0 = \left| \frac{120}{2} \frac{(0)^3}{3} + C_1 \times (0) + C_2 \right|$

$0 = | 0 + 0 + C_2 |$

$\boxed{C_2 = 0}$

condition 2 :- At B, x = 14 m & y = 0

Substituting these values in deflection equation (iii) in all bracket since x = 14 m, we get

$0 = \left| \frac{120}{2} \frac{(14)^3}{3} + C_1 \times 14 + 0 \right| - \left| \frac{120}{2} \frac{(14-3)^3}{3} \right| - \left| \frac{80}{2} \frac{(14-9.5)^3}{3} \right|$

$0 = | 54880 + 14 C_1 | - | 26620 | - | 1215 |$

$= 5480 + 14 C_1 - 26620 - 1215$

$0 = 27045 + 14C_1$

$\therefore -14C_1 = +27045$

$\therefore \boxed{C_1 = -1931.785}$

Step 4 :- To find the deflection at the point D under 80 kN load i.e. $Y_D$.

Let $Y_D$ = Deflection at the point D under 80 kN load

At point D, $x = 9.5$ m from origin A. Hence substituting $x = 9.5$ m and values of $C_1$ and $C_2$ in deflection equation (iii) upto second bracket only. we get,

$$EIy_D = \left| \frac{120}{2}\frac{(9.5)^3}{3} - 1931.785 \times 9.5 + 0 \right| - \left| \frac{120}{2}\frac{(9.5-3)^3}{3} \right|$$

$$\therefore EIy_D = \left| 17147.5 - 18351.9575 \right| - \left| 5492.5 \right|$$

$$\therefore EIy_D = \{ 17147.5 - 18351.9575 - 5492.5$$

$$= -6696.9575$$

$$\therefore y_D = \frac{-6696.9575}{EI}$$

$$= \frac{-6696.9575}{(2.1 \times 10^8) \times (16 \times 10^{-4})}$$

$$= \frac{-6696.1}{33\cancel{6} \times 10^3} = \frac{-6696.9575}{336 \times 10^3}$$

$y_D = -0.01993$ m

$\therefore y_D = -0.1993$ m

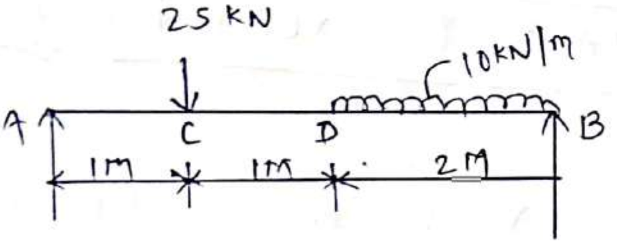
$y_D = -19.93$ mm

$\boxed{y_D = 19.93 \text{ mm}}$ (downward)

(-ve sign indicate downward deflection)

---x---x---

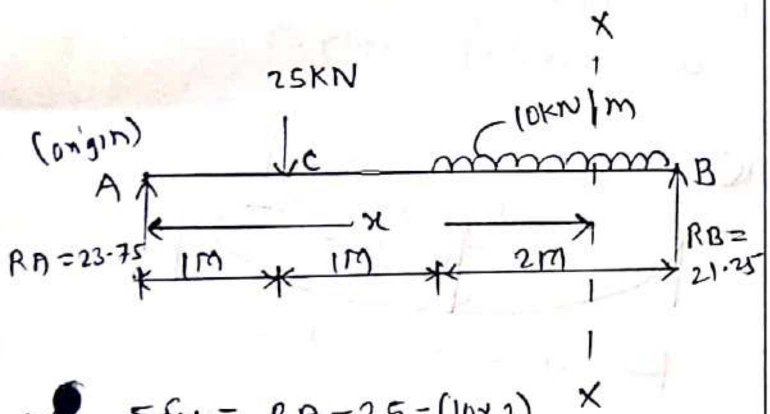
A simply supported beam AB of span 4m is loaded as shown in fig. Determine the slope at A and deflection at midspan. consider $EI = 40,000\ kN/m^2$. Use macaulay's method.




→ Given :- $L = 4m$,

$EI = 40,000\ kN \cdot m^2$

step 1) To find support reactions RA and RB :-




+↑ −↓ $\Sigma fy = RA - 25 - (10 \times 2) + RB$

$0 = RA - 25 - 20 + RB$

$\therefore RA + RB = 45KN \quad \ldots \text{i)}$

$\Sigma M_A = RA \times 0 + 25 \times 1 + (10 \times 2) \times 3) - RB \times 4$

$0 = 25 + 60 - 4RB$

$\therefore \boxed{RB = 21.25KN}$

put in eqn. 1 we get

$RA + 21.25 = 45$

$\boxed{RA = 23.75KN}$

step 2 :- To construct deflection equation by macaulay's method

The general B.M. eqn. at a distance $x$ from A is –

$$EI \frac{d^2y}{dx^2} = Mx$$

$$EI \frac{d^2y}{dx^2} = \left|23.75 \times x\right| - \left|25 \times (x-1)\right| - \left|10 \times (x-2) \times \frac{(x-2)}{2}\right| \quad \text{c) - 2) +}$$

$$= \left|23.75x\right| - \left|25(x-1) - \right.$$
$$\left. - \left|\frac{10}{2}(x-1)^2\right| \right.$$

Integrating above eqn. w.r.t. $x$

$$EI \frac{dy}{dx} = \left|23.75 \frac{x^2}{2} + c_1\right| - \left|\frac{25(x-1)^2}{2}\right| - \left|\frac{10}{2} \frac{(x-2)^3}{3}\right| \ldots \text{slope eqn. } (2)$$

Again integrating w.r.t. x

$$EIY = \left|\frac{23.75}{2} \frac{x^3}{3} + c_1 x + c_2\right| - \left|\frac{25}{2} \frac{(x-1)^3}{3}\right| - \left|\frac{10}{6} \frac{(x-2)^4}{4}\right|$$

--- Deflection eqn. (3)

step 3 Apply boundary conditions to find $c_1$ and $c_2$

condition 1 :- At support A,

$x = 0, \ Y = 0$

put these values in eqn. (3) up to first bracket only

$0 = \left| 0 + c_1 \times 0 + c_2 \right|$

$c_2 = 0$

condition 2 : At support B

$x = 4m,\ y = 0$

put these values in eqn. (3) in all brackets.

$$0 = \left| \frac{23.75\times(4)^3}{2\cdot 3} + c_1\times 4 + 0 \right| - \left| \frac{25}{2}\frac{(4-1)^3}{3} \right| - \left| \frac{10}{6}\frac{(4-2)^4}{4} \right|$$

$$0 = \left| 253.33 + 4c_1 \right| - \left| 112.5 \right| - \left| 6.666 \right|$$

$$0 = 4c_1 + 134.164$$

$$\therefore 4c_1 = -134.164$$

$$\boxed{c_1 = -33.54}$$

* step 4 To find the slope at left support

let, $\theta_A = \left(\frac{dy}{dx}\right)_A$ = slope at left support A

At $x = 0$

put the values of $c_1$ and $x = 0$ in slope eqn. (11) upto first bracket. since at A $x = 0$

$$EI\left(\frac{dy}{dx}\right)_A = \left| \frac{23.75(0)^2}{2} - 33.54 \right|$$

$$EI\left(\frac{dy}{dx}\right)_A = \left| 0 - 33.54 \right|$$

$$EI\left(\frac{dy}{dx}\right)_A = -33.54$$

$$\therefore \left(\frac{dy}{dx}\right)_A = \frac{-33.54}{EI}$$

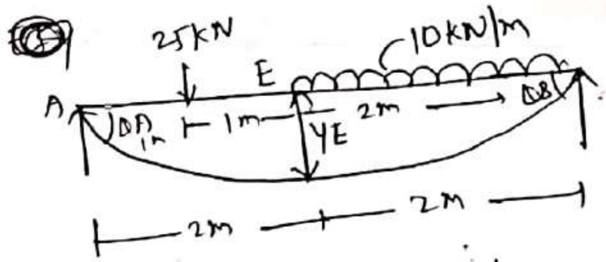
$$\therefore \theta_A = \left(\frac{dy}{dx}\right)_A = \frac{-33.54}{40000}$$

$$\boxed{\theta_A = 8.385\times10^{-4}\ rad}$$ Ans.

* step 5) To find this deflection at mid span

At mid span $x = 2m$

Put the values of $c_1, c_2$ & $x = 2$ m in deflection eqn. (3) upto second bracket only.




Let $y_E$ = defn. at mid span.

$$EIy_E = \left| \frac{23.75}{2}\frac{(2)^3}{3} - 33.54\times2 + 0 \right| - \left| \frac{25}{2}\frac{(2-1)^3}{3} \right|$$

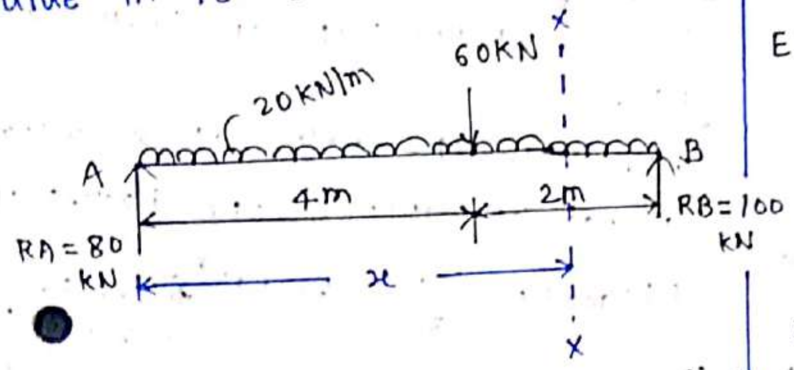
$$= \left| 31.666 - 67.08 \right| - \left| 4.166 \right|$$

$$= \frac{-39.58}{EI} = \frac{-39.58}{40000}$$

$$= -9.895\times10^{-4} m$$

$$= -0.9895 mm \text{ (downward)}$$

A simply supported beam of 6m span carries an udl of 20 KN/m over entire span and a point load of 60KN at 2m from right hand support using macaulay's method, locate the point of maximum deflection and find its value in terms of EI.




Step 1) To calculate support Reaction i.e. RA & RB.

$\uparrow^{+}\downarrow^{-}$ $\Sigma fy = RA - 20\times 6 - 60 + RB$

$0 = RA - 180 + RB$

$RA + RB = 180$ --- i)

$\Sigma M_A = RA\times 0 + (20\times 6)\times\frac{6}{2} + 60\times 4 - RB\times 6$

$0 = 120\times 3 + 240 - 6RB$

$\therefore RB = \frac{600}{6}$ $\therefore RB = 100kN$

substitute the value of RB in eqn. (1)

$RA + 100 = 180$

$\therefore RA = 180 - 100$ $\therefore RA = 80KN$

Step 2) To create slope and deflection eqn. by macaulay's method.

consider a section x-x at a dist. 'x' from origin A in position CB as shown in fig.

$\therefore$ The general bending moment eqn. at a distance 'x' from 'A' is

$$EI\frac{d^2y}{dx^2} = Mx$$

$$= \left|80\times x\right| - \left|20(x)\times\frac{x}{2}\right| - \left|60\times(x-4)\right|$$

$$EI\frac{d^2y}{dx^2} = \left|80x\right| - \left|10x^2\right| - \left|60(x-4)\right|$$

Integrating above eqn. w.r.t 'x'

$$EI\frac{dy}{dx} = \left|80\frac{x^2}{2} + c_1\right| - \left|10\frac{x^3}{3}\right| - \left|60\frac{(x-4)^2}{2}\right| \text{ --- ii)}$$

Integrating eqn. (ii) w.r.t. x

$$EIy = \left|\frac{80}{2}\cdot\frac{x^3}{3} + c_1x + c_2\right| - \left|\frac{10}{3}\cdot\frac{x^4}{4}\right| - \left|\frac{60}{2}\cdot\frac{(x-4)^3}{3}\right|$$

$$\therefore EIy = \left|\frac{40}{3}x^3 + c_1x + c_2\right| - \left|0.833\cdot x^4\right| - \left|10\cdot(x-4)^3\right| \text{ --- iii)}$$

Step 3) Apply the boundary condition to find the values of $c_1$ and $c_2$

* condition 1 :- At A, $x = 0$, $y = 0$

put these values in eqn (iii) upto 1st bracket only, we get.

$0 = |0 + 0 + c_2|$ $\therefore c_2 = 0$

* condition 2 :- At B, $x = 6$, $y = 0$

put these values in eqn (iii) in all bracket, we get

$$0 = \left|\frac{40}{3}(6)^3 + c_1\times 6 + 0\right| - \left|0.833\cdot(6)^4\right| - \left|10\cdot(6-4)^3\right|.$$

$$0 = \left|\begin{matrix}2880\\+6c_1\end{matrix}\right| - \left|1079.56\right| - \left|80\right|$$

$$0 = 2880 + 6c_1 - 1079.56 - 80$$

$0 = 1720.44 + 6C_1$

$\therefore 6C_1 = -1720.44$

$C_1 = \dfrac{-1720.44}{6}$

$\underline{C_1 = -286.74}$

step 4) substitute the value of const $C_1$ and $C_2$ in eqn. (ii) and (iii) we get:

$$EI\frac{dy}{dx} = \left|\frac{80}{2}x^2 - 286.74\right| - \left|\frac{10x^3}{3}\right| - \left|\frac{60(x-4)^2}{2}\right| \quad \text{... slope eq}^n\text{. (iv)}$$

$$EIy = \left|\frac{40}{3}x^3 - 286.74x\right| - \left|0.833x^4\right| - \left|10(x-4)^3\right| \quad \text{... Deflection eq}^n\text{. (v)}$$

step 5) To find Location of maximum deflection.

since the span AC > span CB, the max. deflection occurs in the portion AC. at point of maximum deflection, $\frac{dy}{dx} = 0$.

substitute $\frac{dy}{dx} = 0$ in eqn. (iv) upto second bracket only, we get

$$EI \times 0 = \left|\frac{80}{2}x^2 - 286.74\right| - \left|\frac{10}{3}x^3\right|$$

$$0 = \frac{80}{2}x^2 - 286.74 - \frac{10}{3}x^3$$

$$0 = 40x^2 - 286.74 - 3.33x^3$$

$$3.33x^3 - 40x^2 + 286.67 = 0$$

solving above eqn. for $x$ we get

$x = 11.34m$ or $x = -2.44m$ or $x = 3.109m$

$x = 11.34m$ and $x = -2.44m$ are not valid because $x = 11.34\,m >$ span m

$x = 3.109\,m < 4m$ span AC

Hence $\underline{x = 3.109\,m}$.

* maximum deflection is occured at 3.109 m from A.

* step 6 :- To find maximum deflection in terms of EI :-

substitute the value of $x = 3.109$ m in eqn. (v) upto second bracket only

$$EIy_{max} = \left|\frac{40}{3} \times (3.109)^3 - 286.74 \times (3.109)\right| - \left|0.833 \times (3.109)^4\right|$$

$$= \frac{40}{3} \times (3.109)^3 - 286.74 \times (3.109) - 0.833 \times (3.109)^4$$

$$= 400.68 - 891.47 - 77.82$$

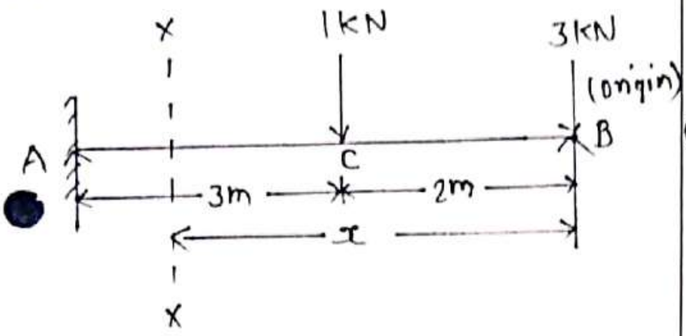
$$EIy_{max} = -568.61$$

$$\therefore \boxed{y_{max} = \frac{-568.61}{EI}} \quad \text{Ans.}$$

* Negative sign indicate downward deflection.

## Cantilever Beam :-

Ex:-1) A cantilever Beam of span 5m carries two concentrated loads one of 1kN at 3m from the fixed end and other 3kN at free end. if the flexural rigidity EI is 8000 kN·m², find slope and deflection under each load.




→ Find $\theta_B$ and $\theta_C$ = ?
$y_B$ and $y_C$ = ?

1) consider section x-x at a dist. x from origin point B.

Now, the governing differential eqn. will be

$$EI\frac{d^2y}{dx^2} = Mx$$

$$EI\frac{d^2y}{dx^2} = -3\cdot x - 1\cdot(x-2) \quad \text{— i)}$$

integrating above eqn. w.r.t. 'x'

$$EI\frac{dy}{dx} = -\frac{3x^2}{2} - \frac{1(x-2)^2}{2} + c_1$$

$$EI\frac{dy}{dx} = -1.5x^2 - 0.5(x-2)^2 + c_1 \quad \text{— 2)}$$

Integrating above eqn. w.r.t. x

$$EIy = \frac{-1.5x^3}{3} - \frac{0.5(x-2)^3}{3} + c_1x + c_2$$

$$EIy = -0.5x^3 - 0.166(x-2)^3 + c_1x + c_2 \quad \text{... 3)}$$

2) to find $c_1$ and $c_2$, use boundary condition:

At point A, $x = 5m$, $\frac{dy}{dx} = 0$ — slope

put in eqn. 2

$$0 = -1.5(5)^2 - 0.5(5-2)^2 + c_1$$

$$0 = -42 + c_1$$

$$\therefore \boxed{c_1 = 42}$$

(fixed end)

At point A, $x = 5m$, $y = 0$ — defln.

put in eqn. 3

$$0 = -0.5(5)^3 - 0.166(5-2)^3 + 42\times5 + c_2$$

$$0 = 143.018 + c_2$$

$$\boxed{c_2 = -143.018}$$

Putting values of $c_1, c_2$ in eqn 2 & 3

$$EI\frac{dy}{dx} = -1.5x^2 - 0.5(x-2)^2 + 42 \quad \text{— 4}$$

$$EIy = -0.5x^3 - 0.166(x-2)^3 + 42x - 143.18 \quad \text{— 5}$$

3) Now to find slopes under each point load, to find $\theta_B$ put $x=0$ in eqn. 4

$$EI\left(\frac{dy}{dx}\right)_B = -1.5\times0^2 - 0.5(0-2)^3 + 42\times0$$

$$EI\,\theta_B = 42$$

$$\boxed{\theta_B = \frac{42}{8000}}$$

$$\theta_B = 5.25\times10^{-3}\,\text{rad}$$

To find $\theta_C$ = put $x = 2m$ in eqn. 4

$$EI\left(\frac{dy}{dx}\right)_C = -1.5\times2^2 - 0.5(2-2)^2 + 42$$

$$EI\,\theta_C = 36$$

$$\theta_C = 36/EI$$

$$\boxed{\theta_C = 4.5 \times 10^{-3} \text{ rad}}$$

4) Now, to find deflections under point load. to find $y_B$ put $x=0$ in eqn. (5)

$$EI \cdot y_B = -0.5 \times 0^3 - 0.166 (0-2)^3 + 42 \times 0 - 143.018$$

$$EI \cdot y_B = -143.018$$

$$y_B = \frac{-143.018}{EI}$$

$$y_B = \frac{-143.018}{8000}$$

$$y_B = 0.01787 \text{ m}$$

$$\boxed{y_B = 17.87 \text{ mm}}$$

* To find $y_C$ = put $x = 2$ in eqn. (5)

$$EI \cdot y_C = -0.5 \times 2^3 - 0.166 \times (2-2)^3 + 42 \times 2 - 143.018$$

$$EI \cdot y_C = -63.018$$

$$y_C = \frac{-63.018}{EI}$$

$$y_C = \frac{-63.018}{8000}$$

$$y_C = -7.877 \times 10^{-3} \text{ m}$$

$$\boxed{y_C = -7.877 \text{m}}$$

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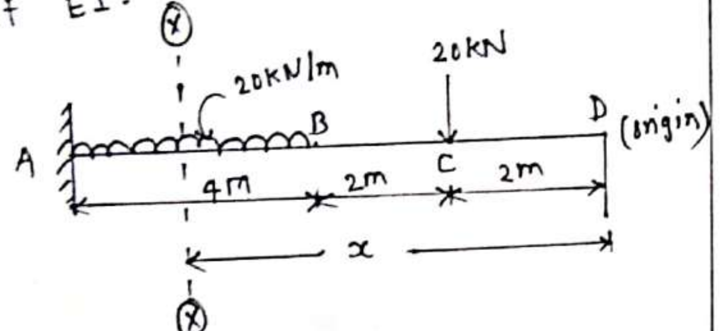
cantilever ABCD is fixed at A and carries a udl of 20 kN/m on portion AB and a point load of 20 kN at C. AB = 4 m, BC = CD = 2m. Find the deflection at B and slope at C in terms of EI.




→ Find :- $y_B = ?$ and $\theta_C = ?$

step 1) Taking section x-x at a distance x from origin, the governing differential equation,

$$EI\cdot\frac{d^2y}{dx^2} = M_x$$

$$EI\cdot\frac{d^2y}{dx^2} = -20\times(x-2) - 20(x-4)\cdot\frac{(x-4)}{2}$$

$$EI\cdot\frac{d^2y}{dx^2} = -20\cdot(x-2) - 10(x-4)^2 \quad \text{---}\rightarrow 1)$$

Integrating above eqn. w.r.t. 'x.'

$$EI\,\frac{dy}{dx} = -\frac{20(x-2)^2}{2} - \frac{10(x-4)^3}{3} + c_1$$

$$EI\cdot\frac{dy}{dx} = -10\cdot(x-2)^2 - 3.33(x-4)^3 + c_1 \rightarrow 2)$$

Integrating above eqn. w.r.t. x again

$$EI\cdot y = -10\cdot\frac{(x-2)^3}{3} - \frac{3.33(x-4)^4}{4} + c_1\cdot x + c_2$$

$$EI\cdot y = -3.33(x-2)^3 - 0.832(x-4)^4 + c_1\cdot x + c_2 \quad \text{--}\rightarrow 3)$$

step 2) To find $c_1$ and $c_2$, use boundary conditions.

At Point A, $x = 8m$, $\frac{dy}{dx} = 0$,

Put in eqn. (2)

$$\therefore 0 = \frac{-10(8-2)^3}{3} - 3.333(8-4)^3 + c_1$$

$$0 = -573.31 + c_1$$

$$\boxed{c_1 = 573.31}$$

At point A, $x = 8m$, $y = 0$, Put in eqn. (3)

$$0 = -3.33(8-2)^3 - 0.832(x-4)^4 + 573.31\times 8 + c_2$$

$$0 = 3654.20 + c_2$$

$$\boxed{c_2 = -3654.20}$$

Put values of $c_1$ and $c_2$ in eqn. (2) and (3),

$$EI\cdot\frac{dy}{dx} = -10(x-2)^2 - 3.33(x-4)^3 + 573.31 \quad \text{---} (4)$$

$$EI\cdot y = -3.33(x-2)^3 - 0.83(x-4)^4 + 573.31x - 3654.20 \quad \text{---} (5)$$

step 3) To find $y_B$, put $x = 4m$ in equation (5)

$$EI\cdot y_B = -3.33(4-2)^3 - 0.83(4-4)^4 + 573.31\times 4 - 3654.20$$

$$EI\,y_B = -1387.6$$

$$\boxed{y_B = \frac{-1387.6}{EI}}$$ Ans.

step 4) To find $\theta_C$, put $x = 2\,m$ in equation (4)

$$EI\left(\frac{dy}{dx}\right)_C = -10(2-2)^2 - 3.33(2-4)^3 + 573.31$$

$$EI\,\theta_C = 573.31$$

$$\boxed{\theta_C = \frac{573.31}{EI}}$$ — Ans.

# Unit-1 Slope and Deflection (Question Bank)

| Que. No. | Question | Marks |
|---|---|---|
| 1 | Define slope and deflection. Differentiate between slope and deflection. | 2 |
| 2 | State the relationship between bending moment, slope and deflection. | 2 |
| 3 | Define flexural rigidity (EI) and stiffness of a beam. | 2 |
| 4 | Explain the Double Integration Method for finding slope and deflection. | 4 |
| 5 | Explain Macaulay's Method. State its advantages over the Double Integration Method. | 4 |
| 6 | Explain the assumptions made in the theory of simple bending used for beam deflection analysis. | 4 |
| 7 | Derive the relation). $EI\frac{d^2y}{dx^2} = M.$ | 4 |
| 8 | A cantilever beam carrying a point load at the free end. Determine the slope and deflection at the free end using the Double Integration Method. | 6 |
| 9 | A cantilever beam carrying a uniformly distributed load over the entire span. Determine the slope and maximum deflection. | 6 |
| 10 | A simply supported beam carrying a concentrated load at the centre. Determine the slope at the supports and maximum deflection. | 6 |
| 11 | A simply supported beam carrying a uniformly distributed load over the entire span. Determine the maximum slope and maximum deflection. | 6 |
| 12 | Determine the slope and deflection of a simply supported beam carrying an eccentric point load using Macaulay's Method. | 6 |
| 13 | Determine the slope and deflection of a cantilever beam carrying two concentrated loads using Macaulay's Method. | 6 |
| 14 | Compare Double Integration Method and Macaulay's Method with suitable points. | 4 |
| 15 | Explain the practical significance of slope and deflection in structural engineering. | 4 |