The Shirpur Education Society's

R. C. Patel College of Engineering and Polytechnic, Shirpur

Department of Applied science and Humanities

NAME OF COURSE: - Basic Mathematics

Unit Name :- Differential Calculus

SUBJECT CODE : - 311302 FY CO/CW/EE/CE/ME

SEMESTER: - FIRST

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Subject :- Basic Mathematics (311302)

Unit : 4 - $\boxed{\text{Differential Calculus}}$

* Function :-

* Intervals -

Interval is a collection of all real numbers which are in between two given real numbers a and b.

1) Open Interval - (a, b)

$$(a, b) = \{x \mid a < x < b,\ a, b \in R\}$$

2) Closed Interval - $[a, b]$

$$[a, b] = \{x \mid a \le x \le b,\ a, b \in R\}$$

3) Semi-open Interval - $(a, b]$

$$(a, b] = \{x \mid a < x \le b,\ a, b \in R\}$$

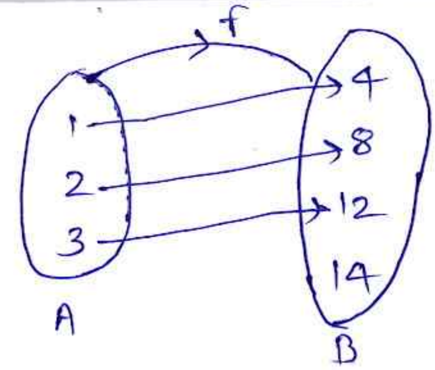
4) Semi-closed Interval - $[a, b)$

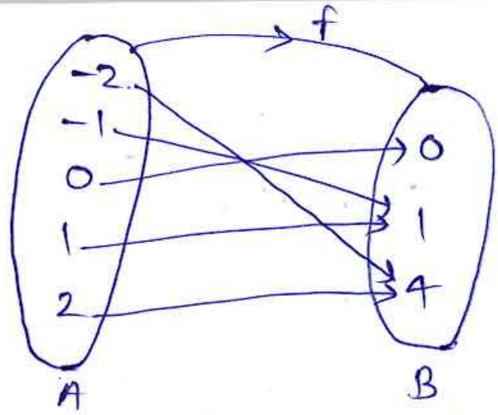
$$[a, b) = \{x \mid a \le x < b,\ a, b \in R\}$$

* Function -

Definition - A function is a relation from set A to set B such that every element of A is related to a unique element of B.
It is denoted by $f : A \longrightarrow B$ OR $A \xrightarrow{f} B$ and read as "f is a function from A to B"
Normally functions are denoted by letters f, g, h etc.

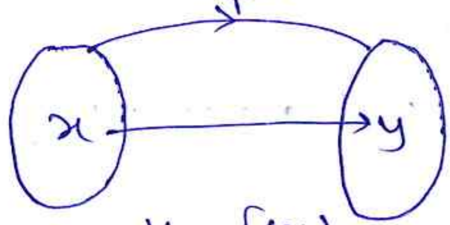
eg.





If $f : A \to B$ is a function and element $x \in A$ is related to $y \in B$ then the write as

$\boxed{y = f(x)}$



$y = f(x)$

where y is called image of x under f.
x is called Preimage of y under f.

* Domain and Range of a function -

Consider a function denoted by $y = f(x)$
Then, the set of value of 'x' is called a domain and set of values of 'y' is called Range or co-domain of function.

eg. $y = f(x) = 3x + 4$, $-5 \le x \le 8$
$f(-5) = 3(-5) + 4 = -15 + 4 = -11$
$f(8) = 3(8) + 4 = 24 + 4 = 28$
Here 'x' can take any value between -5 & 8, including both.
Domain of function $= [-5, 8]$
For each value of x in $[-5, 8]$. We obtain the corresponding value of y which are between -11 & 28, inclusive.
$\therefore$ Range of function $= [-11, 28]$

* Types of function -

1) Explicit Function -

A function $f(x,y)$ is said to be explicit function. If one variable can be expressed as a function of another variable explicitly.

eq. $y = x^2$, $y = \sqrt{x}$

2) Implicit Function -

If the variable x & y can not be separated from each other. i.e. the function in the form $f(x,y) = 0$ is called implicit function.

eq. $x^y - e^{x-y} = 0$ etc.

3) Parametric Function -

Two or more variables can be expressed as a function of one or more variables is called Parametric function. Such a function is of the form.

$$x = f(t) \quad , \quad y = g(t)$$

These functions are called parametric function.

eq. i) $x = \sin t$, $y = \cos t$, when 't' is parameter.

ii) $x = a\cos\theta$, $y = a\sin\theta$, when 'θ' is parameter.

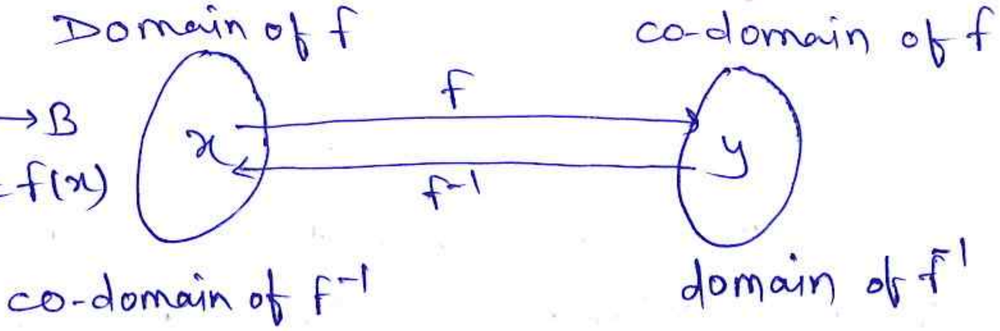
4) Inverse Function -

If y is directly expressed in terms of x, then it may be possible to express x in terms of y. This is called as inverse function. Thus when $y = f(x)$ then its inverse function is $x = f^{-1}(y)$.

If a function $f: A \longrightarrow B$ then inverse function from B to A. It is denoted by $f^{-1}: B \longrightarrow A$

$f: A \to B$

$\therefore y = f(x)$



$f^{-1}: B \to A$

$f^{-1}(y) = x$

5) Even function -

If $f(-x) = f(x)$ then, $f(x)$ is called an even function

eg. $f(x) = x^2$, $f(x) = \cos x$ are even functions.

6) Odd Function -

If $f(-x) = -f(x)$ then $f(x)$ is called an odd function.

eg. $f(x) = x^3$, $f(x) = \sin x$ are odd functions.

7) Rational Function -

The function of the form $f(x) = \dfrac{P(x)}{Q(x)}$, $Q(x) \neq 0$

where $P(x)$ and $Q(x)$ are polynomial in x.

eg. $f(x) = \dfrac{x^2+5x+7}{x+3}$, $f(x) = \dfrac{x^3+1}{x-3}$, if $x \neq 2$

8) Algebraic Function -

A function containing only algebraic operations is called an algebraic function.

eg. $y = x^3 - 2x^2 + 3$, $y = \sqrt{x - \sin x}$ are algebraic function.

9) Composite Function -

If $f(x)$ and $g(x)$ are any two function of x then composition of $f(x)$ and $g(x)$ is denoted by $fog(x)$ and is defined as $fog(x) = f[g(x)]$

eg. $f(x) = x^2$, $g(x) = 3^x$

then $f[g(5)] = f[3^5] = [3^5]^2 = 3^{10}$

$g[f(4)] = g[4^2] = g[16] = 3^{16}$

10) Logarithmic Function -

The functions having logarithm are called logarithmic functions.

e.g. $y = \log_e x$, $y = \log \tan x$, $y = \log_e (x + \sqrt{x^2+1})$

11) Exponential Function -

The functions like a^n, e^x, e^{-n}, a^{-x} are exponential functions.

e.g. $y = a^x$, $y = e^{-x} + e^x$, $y = 5^x + 5^{-x}$

12) Constant Function -

A function $y = f(x)$ is said to be constant function if for different values, the values of $f(x)$ remains constant. It is denoted by $f(x) = k$, k is constant

eg. $f(x) = 7$, for all $x \in R$

13) Transcendental Function -

Functions which are not algebraic are called transcendental functions. Logarithmic, exponential, trigonometric and inverse trigonometric are transcendental functions.

e.g. e^{2x}, $\log x$, $\tan x$, $\sin^{-1} x$ etc.

* $\boxed{\text{Examples}}$:-

Ex.1) If $f(x) = x^2 + 6x + 10$, find $f(2) + f(-2)$

→ Given $f(x) = x^2 + 6x + 10$ —— (i)

Put $x = 2$ in equation (i)

$f(2) = (2)^2 + 6(2) + 10$

$f(2) = 4 + 12 + 10$

$\therefore f(2) = 26$ —————— (ii)

Put $x = -2$ in equation (i)

$f(-2) = (-2)^2 + 6(-2) + 10 = 4 - 12 + 10$

$f(-2) = 2$ —————— (iii)

Adding equations (ii) and (iii)

$f(2) + f(-2) = 26 + 2 = 28$

$\therefore \boxed{f(2) + f(-2) = 28}$

Ex-2) If $f(x) = x^4 - 2x + 7$, find $f(0) + f(2)$.

$\rightarrow$ Given $f(x) = x^4 - 2x + 7$ ——— (i)

Put $x = 0$ in equation (i)

$f(0) = (0)^4 - 2(0) + 7 = 7$

$\therefore f(0) = 7$ ——— (ii)

Put $x = 2$ in equation (i)

$f(2) = (2)^4 - 2(2) + 7$

$= 16 - 4 + 7$

$\therefore f(2) = 19$ ——— (iii)

Adding equations (ii) and (iii)

$f(0) + f(2) = 7 + 19 = 26$

$\therefore \boxed{f(0) + f(2) = 26}$

Ex-3) If $f(x) = x^2 + x + 1$, find $f(x-1)$

$\rightarrow$ Given $f(x) = x^2 + x + 1$ ——— (i)

Put $x = x - 1$ in equation in (i)

$f(x-1) = (x-1)^2 + (x-1) + 1$

$= x^2 - 2x + 1 + x - 1 + 1$

$\therefore \boxed{f(x-1) = x^2 - x + 1}$

Ex-4) If $f(x) = 3x^2 - 5x + 7$ show that $f(-1) = 3f(1)$

$\rightarrow$ Given $f(x) = 3x^2 - 5x + 7$ ——— (i)

Put $x = -1$ in equation (i)

$f(-1) = 3(-1)^2 - 5(-1) + 7 = 3 + 5 + 7$

$\therefore f(-1) = 15$ ——— (ii)

and Put $x = 1$ in equation (i)

$f(1) = 3(1)^2 - 5(1) + 7 = 3 - 5 + 7 = 5$

$\therefore f(1) = 5$ ——— (iii)

$\therefore\ 3f(1) = 3(5)$

$3f(1) = 15$ ——— (iii)

From equations (ii) and (iii)

$$\boxed{f(-1) = 3f(1)}$$

Ex-5) If $f(x) = x^3 - 5x^2 - 4x + 20$ show that $f(0) = -2f(3)$

→ Given $f(x) = x^3 - 5x^2 - 4x + 20$ ——— (i)

Put $x = 0$ in equation (i)

$f(0) = (0)^3 - 5(0)^2 - 4(0) + 20$

$\therefore\ f(0) = 20$ ——— (ii)

and Put $x = 3$ in equation (i)

$f(3) = (3)^3 - 5(3)^2 - 4(3) + 20$

$f(3) = 27 - 45 - 12 + 20$

$\therefore\ f(3) = -10$

$\therefore\ -2f(3) = -2 \times (-10)$

$\therefore\ -2f(3) = 20$ ——— (iii)

From equations (ii) and (iii)

$$\boxed{f(0) = -2f(3)}$$

Ex-6) If $f(x) = x^3 + 1$, find the value of $f(\sqrt[3]{2})$.

→ Given $f(x) = x^3 + 1$

Put $x = \sqrt[3]{2}$ in equation (i)

$\therefore\ f(\sqrt[3]{2}) = (\sqrt[3]{2})^3 + 1$

$= (2^{1/3})^3 + 1$ ——— $[\sqrt[n]{x} = (x)^{1/n}]$

$= 2 + 1$

$$\boxed{f(\sqrt[3]{2}) = 3}$$

Ex-7) If $f(x) = \frac{x^2+1}{x^3-1}$ find $f\left(\frac{1}{2}\right)$

→ Given: $f(x) = \frac{x^2+1}{x^3-1}$ ——— (i)

Put $x = \frac{1}{2}$ in equation (i)

$$f\left(\tfrac{1}{2}\right) = \frac{\left(\frac{1}{2}\right)^2+1}{\left(\frac{1}{2}\right)^3-1} = \frac{\frac{1}{4}+1}{\frac{1}{8}-1} = \frac{\frac{1+4}{4}}{\frac{1-8}{8}} = \frac{\left(\frac{5}{4}\right)}{\left(\frac{-7}{8}\right)}$$

$$f\left(\tfrac{1}{2}\right) = \frac{5}{4} \times \frac{8}{-7} = -\frac{10}{7}$$

$$f\left(\tfrac{1}{2}\right) = -\frac{10}{7} \quad \text{OR} \quad -1.429$$

Ex-8) If $f(x) = 2^x - \log_2 x$ then find $f(2)$

→ Given: $f(x) = 2^x - \log_2 x$ ——— (i)

Put $x = 2$ in equation (i)

$\therefore f(2) = 2^2 - \log_2 2$ $\boxed{\because \log_a a = 1}$

$= 4 - 1$

$\boxed{f(2) = 3}$

Ex-9) If $f(x) = 16^x + \log_4 x$, find $f\left(\frac{1}{2}\right)$

→ Given: $f(x) = 16^x + \log_4 x$ ——— (i)

Put $x = \frac{1}{2}$ in equation (i)

$f\left(\frac{1}{2}\right) = (16)^{\frac{1}{2}} + \log_4\left(\frac{1}{2}\right)$

$= (4^2)^{\frac{1}{2}} - \log_4 2$ $\boxed{\because \log_e\left(\frac{1}{a}\right) = -\log_e a}$

$= 4 - \frac{\log_2 2}{\log_2 4}$ By changing base to 2

$= 4 - \frac{1}{\log_2 (2)^2}$ $\boxed{\because \log_a a = 1}$

$= 4 - \frac{1}{2\log_2 2} = 4 - \frac{1}{2}$

$\therefore \boxed{f\left(\frac{1}{2}\right) = \frac{7}{2}}$

Ex-10) If $f(x) = \log(\sin x)$, find $f(\frac{\pi}{2})$

→ Given: $f(x) = \log(\sin x)$ ——— (i)

Put $x = \frac{\pi}{2}$ in equation (i)

$\therefore f(\frac{\pi}{2}) = \log(\sin\frac{\pi}{2})$

$= \log(\sin 90^\circ)$ $\left[\because \frac{\pi}{2} = \frac{180^\circ}{2} = 90^\circ\right]$

$= \log(1)$

$\boxed{\therefore f(\frac{\pi}{2}) = 0}$ $\left[\begin{array}{l}\because \sin 90^\circ = 1 \\ \because \log 1 = 0\end{array}\right]$

Ex-11) Find 'a' if $f(x) = ax + 10$ and $f(1) = 13$.

→ Given: $f(x) = ax + 10$ ——— (i)

Put $x = 1$ in equation (i)

$\therefore f(1) = a(1) + 10$

But Given $f(1) = 13$

$\therefore 13 = a + 10$

$\therefore a = 13 - 10$

$\therefore \boxed{a = 3}$

Ex-12) If $f(x) = ax^2 + bx + 2$ and $f(1) = 3$, $f(4) = 42$ find a and b.

→ Given: $f(x) = ax^2 + bx + 2$ ——— (i)

Put $x = 1$ in equation (i)

$\therefore f(1) = a(1)^2 + b(1) + 2 = a + b + 2$

But given $f(1) = 3$

$\therefore 3 = a + b + 2$

$\therefore a + b = 1$ ——— (ii)

Now Put $x = 4$ in equation (i)

$f(4) = a(4)^2 + b(4) + 2$

But given $f(4) = 42$

$\therefore 42 = a(16) + b(4) + 2$

$\therefore 16a + 4b = 40$

$\therefore 4(4a + b) = 40$

$\therefore 4a+b = \frac{40}{4}$

$\therefore 4a+b = 10$ ——— (iii)

Solving equation (ii) and (iii)

equation (iii) – equation (ii)

$$\begin{array}{r} 4a+b=10 \\ -\quad \underline{\overset{-}{}a \overset{-}{+} b = \overset{-}{}1} \\ \therefore\ 3a \qquad = 9 \end{array}$$

$\therefore a = \frac{9}{3}$ $\quad \therefore \boxed{a=3}$

Putting the value of a in equation (ii)

$a+b=1$

$3+b=1$

$\therefore b = 1-3$ $\quad \therefore \boxed{b=-2}$

Ex-13) Define odd and even function with suitable example.

→ <u>Even function</u>: If $f(-x) = f(x)$, then $f(x)$ is called even function.

e.g. $f(x) = x^2+1$

$\therefore f(-x) = (-x)^2+1 = x^2+1$

$\therefore f(-x) = f(x)$

<u>Odd function</u>: If $f(-x) = -f(x)$, then $f(x)$ is called odd function.

e.g. $f(x) = x^3+x$

$\therefore f(-x) = (-x)^3+(-x) = -x^3-x$

$f(-x) = -(x^3+x)$

$\therefore f(-x) = -f(x)$

Ex-14) State whether the function $f(x) = \frac{e^x+e^{-x}}{2}$ is odd or even.

→ Given: $f(x) = \frac{e^x+e^{-x}}{2}$ ——— (i)

Put $x=-x$ in equation (i)

$f(-x) = \frac{e^{-x}+e^{-(-x)}}{2} = \frac{e^{-x}+e^{x}}{2} = \frac{e^x+e^{-x}}{2}$

$f(-x) = \frac{e^x+e^{-x}}{2}$

$\therefore f(-x) = f(x)$ ——— [from equation (i)]

$\therefore f(x)$ is an even function.

Ex-15) Whether the function $f(x) = x^3 - 3x + \sin x + x\cos x$ is odd?

$\rightarrow$ Given: $f(x) = x^3 - 3x + \sin x + x\cos x$ ——— (i)

Put $x = -x$ in equation (i)

$\therefore f(-x) = (-x)^3 + 3(-x) + \sin(-x) + (-x)\cos(-x)$

$= -x^3 - 3x - \sin x - x\cos x$ $[\because \cos(-x) = \cos x]$

$= -(x^3 + 3x + \sin x + x\cos x)$

$f(-x) = -f(x)$

$\therefore f(x)$ is an odd function.

Ex-16) If $f(x) = x^2 - 3x + 4$, then solve $f(1-x) = f(2x+1)$.

$\rightarrow$ Given: $f(x) = x^2 - 3x + 4$ ——— (i)

Put $x = 1-x$ in equation (i)

$\therefore f(1-x) = (1-x)^2 - 3(1-x) + 4$

$= 1 - 2x + x^2 - 3 + 3x + 4$

$[\because (a-b)^2 = a^2 - 2ab + b^2]$

$\therefore f(1-x) = x^2 + x + 2$ ——— (ii)

Put $x = 2x+1$ in equation (i)

$f(2x+1) = (2x+1)^2 - 3(2x+1) + 4$

$= 4x^2 + 4x + 1 - 6x - 3 + 4$

$[\because (a+b)^2 = a^2 + 2ab + b^2]$

$f(2x+1) = 4x^2 - 2x + 2$ ——— (iii)

Now given $f(1-x) = f(2x+1)$

$\therefore x^2 + x + 2 = 4x^2 - 2x + 2$ ——— [from equation (ii) & (iii)]

$\therefore x^2 + x = 4x^2 - 2x$

$\therefore 4x^2 - x^2 - 2x - x = 0$

$\therefore 3x^2 - 3x = 0$

$\therefore 3x(x-1) = 0$

$\therefore 3x = 0$ OR $x - 1 = 0$

$\therefore x = 0$ OR $x = 1$

$\therefore \boxed{x = 0}$ OR $\boxed{x = 1}$

Ex-17) If $f(x) = \tan x$, then show that $f(2x) = \frac{2f(x)}{1-[f(x)]^2}$

→ Given: $f(x) = \tan x$ ——— (i)

Put $x = 2x$ in equation (i)

$\therefore f(2x) = \tan(2x)$

$$= \frac{2\tan x}{1-\tan^2 x} = \frac{2\tan x}{1-[\tan x]^2}$$

$$f(2x) = \frac{2f(x)}{1-[f(x)]^2}$$ ——— [from equation (i)]

Ex-18) If $f(x) = \frac{x+3}{4x-5}$ and $t = \frac{3+5x}{4x-1}$ show that $f(t) = x$.

→ Given: $f(x) = \frac{x+3}{4x-5}$ ——— (i)

Put $x = t$ in equation (i)

$$f(t) = \frac{t+3}{4t-5} = \frac{\left(\frac{3+5x}{4x-1}\right)+3}{4\left(\frac{3+5x}{4x-1}\right)-5}$$

$$= \frac{\left(\frac{3+5x}{4x-1}\right)+3}{\left(\frac{12+20x}{4x-1}\right)-5}$$

$$= \frac{\left[\frac{3+5x+3(4x-1)}{(4x-1)}\right]}{\left[\frac{12+20x-5(4x-1)}{(4x-1)}\right]}$$

$$= \frac{3+5x+12x-3}{12+20x-20x+5}$$

$$= \frac{5x+12x}{12+5} = \frac{17x}{17}$$

$\therefore f(t) = x$

Ex-19) If $f(x) = \frac{2x+3}{3x-2}$ Prove that $f[f(x)] = x$.

$\longrightarrow$ Given: $f(x) = \frac{2x+3}{3x-2}$ ——— (i)

Put $x = f(x)$ in equation (i)

$$\therefore f[f(x)] = \frac{2f(x)+3}{3f(x)-2} = \frac{2\left(\frac{2x+3}{3x-2}\right)+3}{3\left(\frac{2x+3}{3x-2}\right)-2}$$

$$= \frac{\frac{4x+6}{3x-2}+3}{\frac{6x+9}{3x-2}-2} = \frac{\left[\frac{4x+6+3(3x-2)}{3x-2}\right]}{\left[\frac{6x+9-2(3x-2)}{3x-2}\right]}$$

$$= \frac{4x+6+9x-6}{6x+9-6x+4} = \frac{4x+9x}{9+4} = \frac{13x}{13}$$

$$\therefore \boxed{f[f(x)] = x}$$

Ex-20) If $f(x) = \log\left(\frac{x}{x-1}\right)$ Show that $f(a+1)+f(a) = \log\left(\frac{a+1}{a-1}\right)$.

$\longrightarrow$ Given: $f(x) = \log\left(\frac{x}{x-1}\right)$ ——— (i)

Put $x = a+1$ in equation (i)

$$f(a+1) = \log\left(\frac{a+1}{a+1-1}\right)$$

$$f(a+1) = \log\left(\frac{a+1}{a}\right) \text{ ——— (ii)}$$

Put $x = a$ in equation (i)

$$f(a) = \log\left(\frac{a}{a-1}\right) \text{ ——— (iii)}$$

Adding equation (ii) and (iii)

$$f(a+1)+f(a) = \log\left(\frac{a+1}{a}\right) + \log\left(\frac{a}{a-1}\right)$$

$$\boxed{\because \log m + \log n = \log(m \times n)}$$

$$= \log\left(\frac{a+1}{a} \times \frac{a}{a-1}\right)$$

$$\therefore \boxed{f(a+1)+f(a) = \log\left(\frac{a+1}{a-1}\right)}$$

Ex-21) If $f(x) = \log(1+\tan x)$, show that $f\left(\frac{\pi}{4}-x\right) = \log 2 - f(x)$.

$\longrightarrow$ Given: $f(x) = \log(1+\tan x)$ ——— (i)

Put $x = \frac{\pi}{4} - x$ in equation (i)

$$f\left(\frac{\pi}{4}-x\right) = \log\left[1+\tan\left(\frac{\pi}{4}-x\right)\right]$$

$$= \log\left[1+\left(\frac{\tan\frac{\pi}{4}-\tan x}{1+\tan\frac{\pi}{4}.\tan x}\right)\right] \quad \left[\because \tan(A-B) = \frac{\tan A - \tan B}{1+\tan A.\tan B}\right]$$

$$= \log\left[1+\left(\frac{1-\tan x}{1+1.\tan x}\right)\right] \quad \left[\because \tan\left(\frac{\pi}{4}\right) = 1\right]$$

$$= \log\left[1+\frac{1-\tan x}{1+\tan x}\right]$$

$$= \log\left[\frac{1+\tan x+1-\tan x}{1+\tan x}\right]$$

$$= \log\left[\frac{2}{1+\tan x}\right] \quad \left[\because \log\left(\frac{m}{n}\right) = \log m - \log n\right]$$

$$= \log 2 - \log(1+\tan x)$$

$$\boxed{f\left(\frac{\pi}{4}-x\right) = \log 2 - f(x)} \text{ ——— [from eq}^n\text{ (i)]}$$

Ex-22) If $f(x) = \frac{3x+2}{4x-3}$ show that $f = f^{-1}$

$\longrightarrow$ Given: $f(x) = \frac{3x+2}{4x-3}$ ——— (i)

Put $x = f(x)$ in equation (i)

$$f[f(x)] = \frac{3f(x)+2}{4f(x)-3}$$

$$= \frac{3\left(\frac{3x+2}{4x-2}\right)+2}{4\left(\frac{3x+2}{4x-3}\right)-3} = \frac{\left(\frac{9x+6}{4x-3}+2\right)}{\left(\frac{12x+8}{4x-3}-3\right)}$$

$$= \frac{\left[\frac{9x+6+2(4x-3)}{4x-3}\right]}{\left[\frac{12x+8-3(4x-3)}{4x-3}\right]} = \frac{9x+6+8x-6}{12x+8-12x+9}$$

$$= \frac{9x+8x}{8+9} = \frac{17x}{17} = x$$

$\therefore f[f(x)] = x$

$\therefore f(x) = f^{-1}(x)$ i.e $\boxed{f = f^{-1}}$

* Limit :-

* Limit of a Function :-

Definition :- Let $f(x)$ be a function of a real variable x. If the difference between $f(x)$ and a fixed value l can be made as small as possible by taking the value of x sufficiently near to a (but not equal to a), then we say that when x tends to a, $f(x)$ tends to l, symbolically, we write $\lim\limits_{x \to a} f(x) = l$.

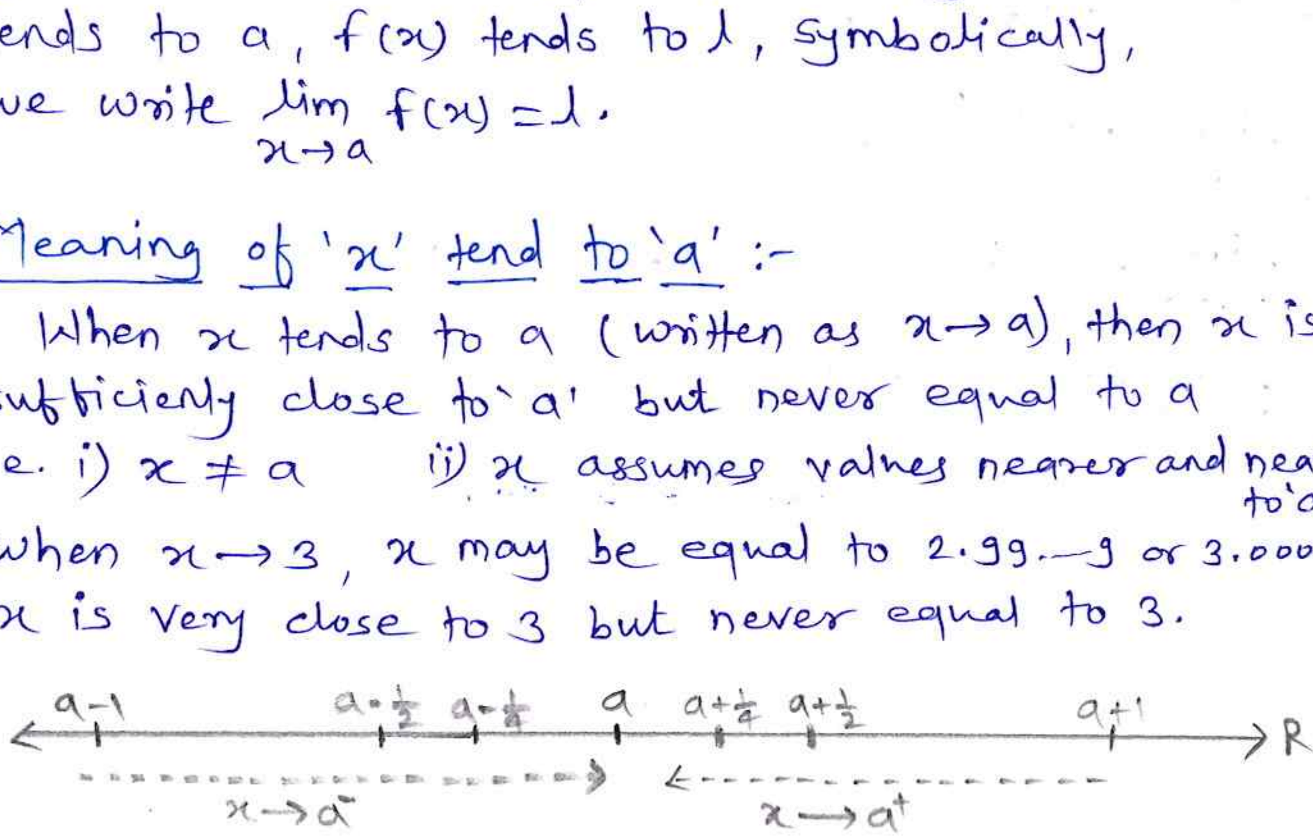
Meaning of 'x' tend to 'a' :-

When x tends to a (written as $x \to a$), then x is sufficiently close to 'a' but never equal to a

i.e. i) $x \neq a$ ii) x assumes values nearer and nearer to 'a'.

e.g. when $x \to 3$, x may be equal to $2.99\ldots9$ or $3.000\ldots$

i.e. x is very close to 3 but never equal to 3.



Algebra of Limits :-

1) $\lim\limits_{x \to a} [f(x) \pm g(x)] = \lim\limits_{x \to a} f(x) \pm \lim\limits_{x \to a} g(x)$

2) $\lim\limits_{x \to a} [f(x) \cdot g(x)] = \lim\limits_{x \to a} f(x) \times \lim\limits_{x \to a} g(x)$

3) $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x \to a} f(x)}{\lim\limits_{x \to a} g(x)}$ only if $\lim\limits_{x \to a} g(x) \neq 0$

Formulae -

1) $\lim\limits_{x \to 0} \dfrac{\sin(x)}{x} = 1$

ii) $\lim\limits_{x \to 0} \cos x = 1$

3) $\lim_{x \to 0} \frac{\tan x}{x} = 1$

4) $\lim_{x \to 0} \frac{e^x - 1}{x} = 1$

5) $\lim_{x \to 0} \frac{x^n - a^n}{x - a} = n a^{n-1}$

6) $\lim_{x \to 0} \frac{a^x - 1}{x} = \log a$

7) $\lim_{x \to 0} (1 + x)^{1/x} = e$

8) $\lim_{x \to 0} \left(1 + \frac{1}{x}\right)^x = e$

9) $\lim_{x \to 0} k = k$ $\qquad$ [k = constant]

10) $\lim_{x \to a} \sin x = \sin a$

* Derivatives :

Derivatives means the rate of change of one variable with respect to other variable.

for eg. 1) The rate of change of displacement with respect to time.

2) The rate of change of Profit with respect to sale.

* **Definition :** If $y = f(x)$ is a function and 'h' is a small change in x and $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ exists then this limit is called as derivative of $f(x)$ w.r.to x or rate of change of $f(x)$ w.r.to x and it is denoted by $f'(x)$.

$$\boxed{f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}}$$

This is also known as the "first principle of derivative".

OR

If $y = f(x)$ is a function of x, δx be a small change in x and $\lim_{\delta x\to 0} \frac{\delta y}{\delta x}$ exists then this limit is called as derivative of y w.r. to x or the rate of change of y with respect to x and it is denoted by $\frac{dy}{dx}$.

$$\therefore \boxed{\frac{dy}{dx} = \lim_{\delta x\to 0} \frac{\delta y}{\delta x}}$$

* **Definition of derivative at a point :-**

The derivative of $y = f(x)$ at $x = a$ is denoted by $f'(a)$ OR $\left[\frac{dy}{dx}\right]_{at\, x=a}$ and it is given by

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

The process of finding derivative is called differentiation.

Notations of Derivative : $\frac{dy}{dx}$, $f'(x)$, y', y_1 etc.

* Derivative of standard Function :-

1) **Constant Function :-**

$$\frac{d}{dx}(k) = 0 \text{ , where } k \text{ is a constant, } k \neq 0$$

2) **Algebraic Function :-**

i) $\frac{d}{dx}(x^n) = n \cdot x^{n-1}$

ii) $\frac{d}{dx}(x) = 1$

iii) $\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$

iv) $\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$

3) **Trigonometric Function :-**

i) $\frac{d}{dx}(\sin x) = \cos x$

ii) $\frac{d}{dx}(\cos x) = -\sin x$

iii) $\frac{d}{dx}(\tan x) = \sec^2 x$

iv) $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

v) $\frac{d}{dx}(\sec x) = \sec x \cdot \tan x$

vi) $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cdot \cot x$

4) **Exponential and Logarithmic function :-**

i) $\frac{d}{dx}(e^x) = e^x$

ii) $\frac{d}{dx}(a^x) = a^x \log a$

iii) $\frac{d}{dx}[\log_e x] = \frac{1}{x}$

5) **Inverse Trigonometric functions :-**

i) $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$

ii) $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$

iii) $\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$

iv) $\frac{d}{dx}(\cot^{-1}x) = \frac{-1}{1+x^2}$

v) $\frac{d}{dx}(\sec^{-1}x) = \frac{1}{x\sqrt{x^2-1}}$

vi) $\frac{d}{dx}(\operatorname{cosec}^{-1}x) = \frac{-1}{x\sqrt{x^2-1}}$

Note :- Derivative of trigonometric and inverse trigonometric function beginning with letter 'c' are Negative.

* Rules of Derivatives :-

* Derivatives of sum or Difference :-

If u and v are differential functions of x and y

$y = u \pm v$ then $\frac{dy}{dx} = \frac{dy}{dx} \pm \frac{dv}{dx}$

Corollary :-

If u, v and w are differentiate function of x,

then $\frac{d}{dx}(u \pm v \pm w) = \frac{d}{dx}(u) \pm \frac{d}{dx}(v) \pm \frac{d}{dx}(w)$

* Examples :-

Ex(1) Find $\frac{dy}{dx}$ if $y = x^{10} + 10^x + e^x$.

→ Given: $y = x^{10} + 10^x + e^x$

Differentiate y w.r.t. x,

$$\frac{dy}{dx} = \frac{d}{dx}(x^{10} + 10^x + e^x) = \frac{d}{dx}(x^{10}) + \frac{d}{dx}(10^x) + \frac{d}{dx}(e^x)$$

$$\left[\because \frac{d}{dx}(x^n) = nx^{n-1},\ \frac{d}{dx}(a^x) = a^x \log a,\ \frac{d}{dx}(e^x) = e^x\right]$$

$$\therefore \frac{dy}{dx} = 10x^{10-1} + 10^x \log 10 + e^x$$

$$\therefore \boxed{\frac{dy}{dx} = 10x^9 + 10^x \log 10 + e^x}$$

Ex-2 Differentiate w.r.t. x : $\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^2$

→ Consider : $y=\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^2$

$\left[\because (a+b)^2 = a^2+2ab+b^2\right]$

$\therefore\ y=(\sqrt{x})^2+2\sqrt{x}\times\frac{1}{\sqrt{x}}+\left(\frac{1}{\sqrt{x}}\right)^2$

$y = x+2+\frac{1}{x}$

Differentiate w.r.t. x

$\frac{dy}{dx}=\frac{d}{dx}(x)+\frac{d}{dx}(2)+\frac{d}{dx}\left(\frac{1}{x}\right)$

$=1+0+\left(\frac{-1}{x^2}\right)$

$\therefore\ \boxed{\frac{dy}{dx}=1-\frac{1}{x^2}}$

Ex-3 find $\frac{dy}{dx}$ if $y=\log x+\log_5 x+\log_5 5$

→ Given : $y=\log x+\log_5 x+\log_5 5$

$y=\log x+\log_5 x+1$ $\quad [\because \log_a a=1]$

Differentiate y w.r.t. x

$\frac{dy}{dx}=\frac{d}{dx}(\log x)+\frac{d}{dx}(\log_5 x)+\frac{d}{dx}(1)$

$=\frac{1}{x}+\frac{d}{dx}\left(\frac{\log x}{\log 5}\right)+0$ $\quad \left[\because \log_y x=\frac{\log x}{\log y}\right]$

$=\frac{1}{x}+\frac{1}{\log 5}\frac{d}{dx}(\log x)$

$\therefore\ \frac{dy}{dx}=\frac{1}{x}+\frac{1}{x\log 5}$

$\therefore\ \boxed{\frac{dy}{dx}=\frac{1}{x}\left[1+\frac{1}{\log 5}\right]}$

Ex-4) Find $\frac{dy}{dx}$ if $y = a^x + x^a + a^a + \sqrt{x}$

$\longrightarrow$ Given : $y = a^x + x^a + a^a + \sqrt{x}$

Differentiate y w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dx}(a^x) + \frac{d}{dx}(x^a) + \frac{d}{dx}(a^a) + \frac{d}{dx}(\sqrt{x})$$

$$= a^x \cdot \log a + a \cdot x^{a-1} + 0 + \frac{1}{2\sqrt{x}}$$

$$\therefore \boxed{\frac{dy}{dx} = a^x \log a + a \cdot x^{a-1} + \frac{1}{2\sqrt{x}}}$$

Ex-(5) Find $\frac{dy}{dx}$ if $y = (3a)^x + x^{\log 3} + x^a + a^a$

$\longrightarrow$ Given $y = (3a)^x + x^{\log 3} + x^a + a^a$

Differentiate w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dx}(3a)^x + \frac{d}{dx}(x^{\log 3}) + \frac{d}{dx}(x^a) + \frac{d}{dx}(a^a)$$

$$= (3a)^x \cdot \log(3a) + \log 3 \times x^{(\log 3)-1} + a \cdot x^{a-1} + 0$$

$\left[\because \text{Using } \frac{d}{dx}(a^x) = a^x \log a \;,\; \frac{d}{dx}(x^n) = n \cdot x^{n-1}\right]$

$$\therefore \boxed{\frac{dy}{dx} = 3^x \cdot a^x \cdot \log(3a) + \log 3 \cdot x^{(\log 3)-1} + a \cdot x^{a-1}}$$

Ex-(6) Find $\frac{dy}{dx}$ if $y = 3\sin x + \tan x$.

$\longrightarrow$ Given : $y = 3\sin x + \tan x$

Differentiate w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dx}(3\sin x) + \frac{d}{dx}(\tan x)$$

$$= 3\frac{d}{dx}(\sin x) + \sec^2 x$$

$$\therefore \boxed{\frac{dy}{dx} = 3\cos x + \sec^2 x}$$

* Derivative of Product :-

If u and v are differentiable functions of x and $y = u \cdot v$ then $\boxed{\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}}$

* Corollary - 1 :

If $y = cu$ where c is a constant and y and u are functions of x then, $\boxed{\frac{dy}{dx} = c\frac{du}{dx}}$

* Corollary : 2 :

If $y = uvw$, where u, v and w are function of x then $\boxed{\frac{dy}{dx} = uv\frac{dw}{dx} + uw\frac{dv}{dx} + vw\frac{du}{dx}}$

* Examples :-

Ex-1) Find $\frac{dy}{dx}$ if $y = x \cdot e^x$

→ Given: $y = x \cdot e^x$

Differentiate w.r.t. x

$$\frac{dy}{dx} = x\frac{d}{dx}(e^x) + e^x\frac{d}{dx}(x)$$

$\left[\because \text{Product Rule} \quad \frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}\right]$

$$\therefore \frac{dy}{dx} = x \cdot e^x + e^x(1)$$

$$\therefore \boxed{\frac{dy}{dx} = e^x(x+1)}$$

Ex-2) If $y = e^x \cdot \sin x$, find $\frac{dy}{dx}$

→ Given: $y = e^x \cdot \sin x$

Differentiate w.r.t. x

$$\frac{dy}{dx} = e^x\frac{d}{dx}(\sin x) + \sin x\frac{d}{dx}(e^x)$$

$$= e^x \cdot \cos x + \sin x \cdot e^x$$

$$\therefore \boxed{\frac{dy}{dx} = e^x(\cos x + \sin x)}$$

Ex-3) Differentiate $\sin x . \cos x$ w.r.t. x.

$\longrightarrow$ Let $y = \sin x . \cos x$

Differentiate y w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dx}(\sin x . \cos x)$$

$$= \sin x \frac{d}{dx}(\cos x) + \cos x \frac{d}{dx}(\sin x)$$

$$= \sin x(-\sin x) + \cos x(\cos x)$$

$$= \cos^2 x - \sin^2 x$$

$$\therefore \boxed{\frac{dy}{dx} = \cos(2x)} \qquad [\because \cos(2\theta) = \cos^2\theta - \sin^2\theta]$$

Ex-4) If $y = e^x . \sin x . \cos x$, find $\frac{dy}{dx}$.

$\longrightarrow$ Given: $y = e^x . \sin x . \cos x$

Differentiate y w.r.t. x

$$\frac{dy}{dx} = e^x \sin x \frac{d}{dx}(\cos x) + e^x \cos x \frac{d}{dx}(\sin x) + \sin x . \cos x \frac{d}{dx}(e^x)$$

$$= e^x \sin x(-\sin x) + e^x \cos x . \cos x + \sin x . \cos x . e^x$$

$$\therefore \boxed{\frac{dy}{dx} = e^x[-\sin^2 x + \cos^2 x + \sin x . \cos x]}$$

Ex-5) Differentiate $x \tan x - \cos x$ w.r.t. x.

$\longrightarrow$ Let $y = x . \tan x - \cos x$

Differentiate y w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dx}(x . \tan x) - \frac{d}{dx}(\cos x)$$

$$= x \frac{d}{dx}(\tan x) + \tan x \frac{d}{dx}(x) - \frac{d}{dx}(\cos x)$$

$$= x \sec^2 x + \tan x (1) - (-\sin x)$$

$$\therefore \boxed{\frac{dy}{dx} = x \sec^2 x + \tan x + \sin x}$$

Ex-6) Find $\frac{dy}{dx}$ if $y = e^x \cdot \sin^{-1}x$.

$\longrightarrow$ Given $y = e^x \sin^{-1}x$

Differentiate y w.r.t. x

$$\frac{dy}{dx} = e^x \frac{d}{dx}(\sin^{-1}x) + \sin^{-1}x \frac{d}{dx}(e^x)$$

$$= e^x \cdot \frac{1}{\sqrt{1-x^2}} + \sin^{-1}x \cdot e^x$$

$$\therefore \boxed{\frac{dy}{dx} = e^x\left[\frac{1}{\sqrt{1-x^2}} + \sin^{-1}x\right]}$$

* **Derivative of Quotient :-**

If u and v are differentiate function of x and $y = \frac{u}{v}$ where $v \neq 0$ then

$$\therefore \boxed{\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}}$$

* **Corollary :-**

If $y = \frac{1}{v}$ then $\boxed{\frac{dy}{dx} = \frac{-1}{v^2}\frac{dv}{dx}}$

* **Examples :-**

Ex-1) If $y = \frac{\log x}{x}$ find $\frac{dy}{dx}$.

$\longrightarrow$ Given: $y = \frac{\log x}{x}$

Differentiate y w.r.t. x

$$\frac{dy}{dx} = \frac{x\frac{d}{dx}(\log x) - \log x\frac{d}{dx}(x)}{x^2} \quad [\because \text{By Quotient Rule}]$$

$$\frac{dy}{dx} = \frac{x \times \frac{1}{x} - \log x \times 1}{x^2} \quad \left[\because \frac{d}{dx}(\log x) = \frac{1}{x}, \frac{d}{dx}(x) = 1\right]$$

$$\therefore \boxed{\frac{dy}{dx} = \frac{1 - \log x}{x^2}}$$

Ex-2) Find $\frac{dy}{dx}$ if $y = \frac{e^x+1}{e^x-1}$

$\longrightarrow$ Given: $y = \frac{e^x+1}{e^x-1}$

Differentiate y w.r.t. x

$$\frac{dy}{dx} = \frac{(e^x-1)\frac{d}{dx}(e^x+1) - (e^x+1)\frac{d}{dx}(e^x-1)}{(e^x-1)^2}$$

$$= \frac{(e^x-1)\left[\frac{d}{dx}(e^x) + \frac{d}{dx}(1)\right] - (e^x+1)\left[\frac{d}{dx}(e^x) - \frac{d}{dx}(1)\right]}{(e^x-1)^2}$$

$$= \frac{(e^x-1)(e^x+0) - (e^x+1)(e^x-0)}{(e^x-1)^2}$$

$$\frac{dy}{dx} = \frac{(e^x-1)e^x - (e^x+1)e^x}{(e^x-1)^2}$$

Taking e^x common in numerator

$$\frac{dy}{dx} = \frac{e^x[(e^x-1) - (e^x+1)]}{(e^x-1)^2} = \frac{e^x[e^x-1-e^x-1]}{(e^x-1)^2}$$

$$= \frac{e^x[-2]}{(e^x-1)^2}$$

$$\therefore \boxed{\frac{dy}{dx} = \frac{-2e^x}{(e^x-1)^2}}$$

Ex-3) Find $\frac{dy}{dx}$ if $y = \frac{5e^x}{3e^x+1}$ at $x=0$.

$\longrightarrow$ Given $y = \frac{5e^x}{3e^x+1}$

Differentiate y w.r.t. x

$$\frac{dy}{dx} = \frac{(3e^x+1)\frac{d}{dx}(5e^x) - 5e^x\frac{d}{dx}(3e^x+1)}{(3e^x+1)^2}$$

$$\frac{dy}{dx} = \frac{(3e^x+1)\,5.e^x - 5.e^x(3e^x}{(3e^x+1)^2}$$

$$= \frac{15.e^{2x} + 5e^x - 15.e^{2x}}{(3e^x+1)^2}$$

$$\frac{dy}{dx} = \frac{5.e^x}{(3e^x+1)^2}$$

at $x = 0$

$$\frac{dy}{dx} = \frac{5.e^0}{(3e^0+1)^2} = \frac{5}{(3+1)^2} = \frac{5}{(4)^2} = \frac{5}{16} \text{ OR } 0.3125$$

* **Derivative of Composite Function :-**

If 'y' is a function of u and u is a function of x, then $\boxed{\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}}$

This is known as chain rule and it is useful for finding derivatives of composite function.

* **Corollary :-**

If 'y' is a function of 'u', 'u' is a function of 'v', 'v' is a function of 'w' and 'w' is a function of 'x', then $\boxed{\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dv} \times \frac{dv}{dw} \times \frac{dw}{dx}}$

* **Examples :-**

Ex-1) Find $\frac{dy}{dx}$ if $y = \sin(2x+1)$

→ Given: $y = \sin(2x+1)$

Differentiate y w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dx}\sin(2x+1)$$

$$= \cos(2x+1)\,\frac{d}{dx}(2x+1)$$

$$\therefore \frac{dy}{dx} = \cos(2x+1)\left[\frac{d}{dx}(2x) + \frac{d}{dx}(1)\right] \quad \left[\because \frac{d}{dx}\sin[f(x)] = \cos[f(x)]\,\frac{d}{dx}f(x)\right]$$

$$\therefore \frac{dy}{dx} = \cos(2x+1)\,(2+0)$$

$$\therefore \boxed{\frac{dy}{dx} = 2\cos(2x+1)}$$

Ex-2) Find $\frac{dy}{dx}$ if $y = \tan(4-3x)$.

$\rightarrow$ Given: $y = \tan(4-3x)$

Differentiate y w.r.t. x

$$\frac{dy}{dx} = \sec^2(4-3x)\,\frac{d}{dx}(4-3x)$$

$$= \sec^2(4-3x)\left[\frac{d}{dx}(4) - \frac{d}{dx}(3x)\right]$$

$$= \sec^2(4-3x)\,(0-3)$$

$$\therefore \boxed{\frac{dy}{dx} = -3\sec^2(4-3x)}$$

Ex-3) If $y = e^{3\sec x + 4\tan x}$ find $\frac{dy}{dx}$.

$\rightarrow$ Given: $y = e^{3\sec x + 4\tan x}$

Differentiate y w.r.t. x

$$\frac{dy}{dx} = e^{3\sec x + 4\tan x}\,\frac{d}{dx}(3\sec x + 4\tan x)$$

$$= e^{3\sec x + 4\tan x}\left[3\frac{d}{dx}(\sec x) + 4\frac{d}{dx}(\tan x)\right]$$

$$= e^{3\sec x + 4\tan x}\left[3\sec x.\tan x + 4\sec^2 x\right]$$

$$\therefore \frac{dy}{dx} = e^{3\sec x + 4\tan x}\,\sec x\left[3\tan x + 4\sec x\right]$$

Ex-4) If $y = 2e^{3x} + \tan x - \cos(2x) + 9\sin^{-1}x$ find $\frac{dy}{dx}$.

$\rightarrow$ Given: $y = 2.e^{3x} + \tan x - \cos(2x) + 9\sin^{-1}x$

Differentiate y w.r.t. x

$$\frac{dy}{dx} = 2\frac{d}{dx}(e^{3x}) + \frac{d}{dx}(\tan x) - \frac{d}{dx}\cos(2x) + 9\frac{d}{dx}(\sin^{-1}x)$$

$$\frac{dy}{dx} = 2.e^{3x}\frac{d}{dx}(3x) + \sec^2 x - [-\sin(2x)]\frac{d}{dx}(2x) + 9\times\frac{1}{\sqrt{1-x^2}}$$

$$\frac{dy}{dx} = 2 \cdot e^{3x} \cdot 3 + \sec^2 x + 2\sin(2x) + \frac{9}{\sqrt{1-x^2}}$$

$$\therefore \boxed{\frac{dy}{dx} = 6e^{3x} + \sec^2 x + 2\sin(2x) + \frac{9}{\sqrt{1-x^2}}}$$

Ex-5) If $y = \sin x \cdot \cos(2x)$ find $\frac{dy}{dx}$

$\longrightarrow$ Given: $y = \sin x \cdot \cos(2x)$

Differentiate y w.r.t. x

$$\frac{dy}{dx} = \sin x \frac{d}{dx}[\cos(2x)] + \cos(2x)\frac{d}{dx}[\sin x]$$

$$= \sin x[-\sin(2x)]\frac{d}{dx}(2x) + \cos(2x) \cdot \cos x$$

$$\therefore \boxed{\frac{dy}{dx} = -2\sin x \cdot \sin(2x) + \cos(2x) \cdot \cos x}$$

Ex-6) Find $\frac{dy}{dx}$ if $y = \log(x^2+2x+5)$

$\longrightarrow$ Given: $y = \log(x^2+2x+5)$

Differentiate y w.r.t. x

$$\frac{dy}{dx} = \frac{1}{x^2+2x+5} \cdot \frac{d}{dx}(x^2+2x+5)$$

$$= \frac{1}{x^2+2x+5}\left[\frac{d}{dx}(x^2) + \frac{d}{dx}(2x) + \frac{d}{dx}(5)\right]$$

$$= \frac{1}{x^2+2x+5}[2x+2+0]$$

$$\therefore \boxed{\frac{dy}{dx} = \frac{2(x+1)}{x^2+2x+5}}$$

Ex-7) If $y = \log(x^2 \cdot e^x)$, find $\frac{dy}{dx}$

$\longrightarrow$ Given: $y = \log(x^2 \cdot e^x)$

Differentiate y w.r.t. x

$$\frac{dy}{dx} = \frac{1}{x^2 \cdot e^x}\frac{d}{dx}(x^2 \cdot e^x)$$

$$= \frac{1}{x^2 \cdot e^x}\left[x^2\frac{d}{dx}(e^x) + e^x\frac{d}{dx}(x^2)\right]$$

$$\therefore \frac{dy}{dx} = \frac{1}{x^2 \cdot e^x}\left[x^2 \cdot e^x + e^x \cdot 2x\right]$$

$$= \frac{1}{x^2 \cdot e^x}\, x e^x (x+2)$$

$$\therefore \boxed{\frac{dy}{dx} = \frac{x+2}{x}}$$

Ex-8) Find $\frac{dy}{dx}$ if $y = \log\left(\frac{\sin x}{1+\cos x}\right)$.

$\longrightarrow$ Given: $y = \log\left(\frac{\sin x}{1+\cos x}\right)$

$$y = \log(\sin x) - \log(1+\cos x)$$

Differentiate y w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dx}\log(\sin x) - \frac{d}{dx}(1+\cos x)$$

$$= \frac{1}{\sin x} \times \cos x - \frac{1}{1+\cos x}(0 - \sin x)$$

$$= \frac{\cos x}{\sin x} + \frac{\sin x}{1+\cos x}$$

$$= \frac{\cos x(1+\cos x) + \sin^2 x}{\sin x\,(1+\cos x)}$$

$$= \frac{\cos x + \cos^2 x + \sin^2 x}{\sin x\,(1+\cos x)}$$

$$= \frac{1+\cos x}{\sin x\,(1+\cos x)}$$

$$= \frac{1}{\sin x}$$

$$\therefore \boxed{\frac{dy}{dx} = \operatorname{cosec} x}$$

Ex-9) If $y = \sqrt{\frac{1-\cos(2x)}{1+\cos(2x)}}$, find $\frac{dy}{dx}$.

$\longrightarrow$ Given: $y = \sqrt{\frac{1-\cos(2x)}{1+\cos(2x)}}$

$$y = \sqrt{\frac{2\sin^2 x}{2\cos^2 x}}$$

$$y = \sqrt{\tan^2 x}$$

$$y = \tan x$$

Differentiate w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dx}(\tan x)$$

$$\therefore \boxed{\frac{dy}{dx} = \sec^2 x}$$

* Derivatives of Inverse Trigonometric Functions :-

1) $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$

2) $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$

3) $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$

4) $\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$

5) $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$

6) $\frac{d}{dx}(\operatorname{cosec}^{-1} x) = \frac{-1}{x\sqrt{x^2-1}}$

* Properties of Inverse Trigonometric Functions :-

1) $\sin^{-1}(\sin \theta) = \theta$	1) $\sin(\sin^{-1} x) = x$
2) $\cos^{-1}(\cos \theta) = \theta$	2) $\cos(\cos^{-1} x) = x$
3) $\tan^{-1}(\tan \theta) = \theta$	3) $\tan(\tan^{-1} x) = x$
4) $\cot^{-1}(\cot \theta) = \theta$	4) $\cot(\cot^{-1} x) = x$
5) $\sec^{-1}(\sec \theta) = \theta$	5) $\sec(\sec^{-1} x) = x$
6) $\operatorname{cosec}^{-1}(\operatorname{cosec} \theta) = \theta$	6) $\operatorname{cosec}(\operatorname{cosec}^{-1} x) = x$

7) $\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$ when $x, y > 0$ and $xy < 1$

8) $\tan^{-1} x + \tan^{-1} y = \pi + \tan^{-1}\left(\frac{x+y}{1-xy}\right)$ when $x, y > 0$ & $xy = 1$

9) $\tan^{-1} x - \tan^{-1} y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$ when $x > 0, y > 0$

* Substitution Method :-

Sometimes before differentiation, we reduce the given function in a simple form by suitable substitution.

* Standard substitution for finding derivative of inverse Trigonometric function :-

Expression	Standard Substitution	Trigonometric formulae
$2x\sqrt{1-x^2}$	$x = \sin\theta$ or $x = \cos\theta$	$\sin(2\theta) = 2\sin\theta\cos\theta$
$\dfrac{2x}{1+x^2}$	$x = \tan\theta$	$\sin(2\theta) = \dfrac{2\tan\theta}{1+\tan^2\theta}$
$1-2x^2$	$x = \sin\theta$	$\cos(2\theta) = 1-2\sin^2\theta$
$2x^2-1$	$x = \cos\theta$	$\cos(2\theta) = 2\cos^2\theta - 1$
$\dfrac{1-x^2}{1+x^2}$	$x = \tan\theta$	$\cos(2\theta) = \dfrac{1-\tan^2\theta}{1+\tan^2\theta}$
$\dfrac{2x}{1-x^2}$	$x = \tan\theta$	$\tan(2\theta) = \dfrac{2\tan\theta}{1-\tan^2\theta}$
$3x-4x^3$	$x = \sin\theta$	$\sin(3\theta) = 3\sin\theta - 4\sin^3\theta$
$\dfrac{3x-x^3}{1-3x^2}$	$x = \tan\theta$	$\tan(3\theta) = \dfrac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}$
$4x^3-3x$	$x = \tan\theta$	$\cos(3\theta) = 4\cos^3\theta - 3\cos\theta$
$\sqrt{1+x^2}+x$	$x = \tan\theta$	$1+\tan^2\theta = \sec^2\theta$
$\dfrac{1}{\sqrt{1+x^2}}$	$x = \tan\theta$	$1+\tan^2\theta = \sec^2\theta$
$\dfrac{x}{\sqrt{1-x^2}}$ OR $\dfrac{\sqrt{1-x^2}}{x}$	$x = \sin\theta$ OR $x = \cos\theta$	$\cos^2\theta = 1-\sin^2\theta$ OR $\sin^2\theta = 1-\cos^2\theta$
$\sqrt{\dfrac{1+x}{2}}$	$x = \cos\theta$	$1+\cos\theta = 2\cos^2\left(\dfrac{\theta}{2}\right)$

Ex-1) If $y = \cos^{-1}(\sin x)$, find $\frac{dy}{dx}$.

→ Given: $y = \cos^{-1}(\sin x)$

$\therefore \quad y = \cos^{-1}\left[\cos\left(\frac{\pi}{2} - x\right)\right]$

$\therefore \quad y = \frac{\pi}{2} - x \qquad \left[\because \cos^{-1}(\cos\theta) = \theta\right]$

Differentiate y w.r.t. x

$\therefore \quad \frac{dy}{dx} = 0 - 1$

$\therefore \quad \boxed{\frac{dy}{dx} = -1}$

Ex-2) If $y = \sin^{-1}[3x - 4x^3]$, find $\frac{dy}{dx}$.

→ Given: $y = \sin^{-1}[3x - 4x^3]$

Put $x = \sin\theta \qquad \therefore \theta = \sin^{-1} x$ — ①

$y = \sin^{-1}[3\sin\theta - 4\sin^3\theta]$

$y = \sin^{-1}[\sin(3\theta)] \qquad [\because \sin(3\theta) = 3\sin\theta - 4\sin^3\theta]$

$y = 3\theta$

$\therefore \quad y = 3\sin^{-1} x$ — from ①

Differentiate y w.r.t. x

$\frac{dy}{dx} = 3\frac{d}{dx}(\sin^{-1}x) = 3 \times \frac{1}{\sqrt{1-x^2}}$

$\therefore \quad \boxed{\frac{dy}{dx} = \frac{3}{\sqrt{1-x^2}}}$

Ex-3) If $y = \sin^{-1}\left[\frac{1}{\sqrt{1+x^2}}\right]$, find $\frac{dy}{dx}$.

→ Given: $y = \sin^{-1}\left[\frac{1}{\sqrt{1+x^2}}\right]$

Put $x = \tan\theta \qquad \therefore \theta = \tan^{-1} x$ — ①

$\therefore \quad y = \sin^{-1}\left[\frac{1}{\sqrt{1+\tan^2\theta}}\right] \qquad [\because 1 + \tan^2\theta = \sec^2\theta]$

$\therefore \quad y = \sin^{-1}\left[\frac{1}{\sqrt{\sec^2\theta}}\right]$

$\therefore \quad y = \sin^{-1}\left[\frac{1}{\sec\theta}\right]$

$\therefore \quad y = \sin^{-1}(\cos\theta)$

$\therefore \quad y = \sin^{-1}\left[\sin\left(\frac{\pi}{2}-\theta\right)\right] \qquad \left[\because \sin\left(\frac{\pi}{2}-\theta\right) = \cos\theta\right]$

$\therefore \quad y = \frac{\pi}{2} - \theta$

$\therefore \quad y = \frac{\pi}{2} - \tan^{-1}x$ —— from eq^n (1)

Differentiate y w.r.t. x

$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}\left(\frac{\pi}{2}\right) - \frac{d}{dx}\left(\tan^{-1}x\right)$

$\frac{dy}{dx} = 0 - \frac{1}{1+x^2}$

$\therefore \quad \boxed{\frac{dy}{dx} = \frac{-1}{1+x^2}}$

Ex-4) Differentiate w.r.t. x $\tan^{-1}\left(\frac{5x}{1-6x^2}\right)$.

$\longrightarrow$ Let $\quad y = \tan^{-1}\left(\frac{5x}{1-6x^2}\right)$

$y = \tan^{-1}\left[\frac{2x+3x}{1-(2x)\times(3x)}\right]$

Put $\tan A = 2x$ and $\tan B = 3x$

$\therefore \quad A = \tan^{-1}(2x)$ and $B = \tan^{-1}(3x)$ —— (1)

$y = \tan^{-1}\left[\frac{\tan A + \tan B}{1 - \tan A \times \tan B}\right]$

$= \tan^{-1}\left[\tan(A+B)\right]$

$= A + B \qquad \left[\because \tan^{-1}(\tan\theta) = \theta\right]$

$y = \tan^{-1}(2x) + \tan^{-1}(3x)$ —— from eq^n (1)

Differentiate y w.r.t. x

$$\therefore \frac{dy}{dx} = \frac{d}{dx}\tan^{-1}(2x) + \frac{d}{dx}\tan^{-1}(3x)$$

$$= \frac{1}{1+(2x)^2}\cdot\frac{d}{dx}(2x) + \frac{1}{1+(3x)^2}\cdot\frac{d}{dx}(3x)$$

$$= \frac{1}{1+4x^2}(2) + \frac{1}{1+9x^2}(3)$$

$$\therefore \boxed{\frac{dy}{dx} = \frac{2}{1+4x^2} + \frac{3}{1+9x^2}}$$

Ex-5) If $y = \tan^{-1}\left[\frac{a+x}{1-ax}\right]$, find $\frac{dy}{dx}$.

$\longrightarrow$ Given : $y = \tan^{-1}\left[\frac{a+x}{1-ax}\right]$

Put $\tan A = a$ and $\tan B = x$

$\therefore \quad A = \tan^{-1}a$ and $B = \tan^{-1}x$ —— ①

$$y = \tan^{-1}\left[\frac{\tan A + \tan B}{1 - \tan A\,.\tan B}\right]$$

$$y = \tan^{-1}[\tan(A+B)]$$

$$y = A + B$$

$\therefore \quad y = \tan^{-1}a + \tan^{-1}x$ —— from eqⁿ ①

Differentiate y w.r.t. x

$$\therefore \frac{dy}{dx} = \frac{d}{dx}\tan^{-1}a + \frac{d}{dx}\tan^{-1}x$$

$$= 0 + \frac{1}{1+x^2}$$

$$\therefore \boxed{\frac{dy}{dx} = \frac{1}{1+x^2}}$$

Ex-6) Find $\frac{dy}{dx}$ if $y = \tan^{-1}\left[\frac{x}{\sqrt{1-x^2}}\right]$

$\longrightarrow$ Given : $y = \tan^{-1}\left[\frac{x}{\sqrt{1-x^2}}\right]$

Put $x = \sin\theta$, $\theta = \sin^{-1}x$ —— ①

$\therefore\ y = \tan^{-1}\left[\frac{\sin\theta}{\sqrt{1-\sin^2\theta}}\right]$

$\therefore\ y = \tan^{-1}\left[\frac{\sin\theta}{\sqrt{\cos^2\theta}}\right]$

$\therefore\ y = \tan^{-1}\left[\frac{\sin\theta}{\cos\theta}\right]$

$\therefore\ y = \tan^{-1}(\tan\theta)$

$\therefore\ y = \theta$

$\therefore\ y = \sin^{-1}x$ —— from eqⁿ ①

Differentiate y w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dx}(\sin^{-1}x)$$

$$\therefore\ \boxed{\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}}$$

Ex-7) Find $\frac{dy}{dx}$ if $y = \sec^{-1}\left[\frac{1}{4x^3-3x}\right]$

→ Given: $y = \sec^{-1}\left[\frac{1}{4x^3-3x}\right]$

Put $x = \cos\theta$, $\theta = \cos^{-1}x$ —— ①

$$\therefore\ y = \sec^{-1}\left[\frac{1}{4\cos^3\theta - \cos\theta}\right]$$

$$= \sec^{-1}\left[\frac{1}{\cos(3\theta)}\right]$$

$$= \sec^{-1}[\sec(3\theta)]$$

$$y = 3\theta$$

$\therefore\ y = 3\cos^{-1}x$ —— from eqⁿ ①

Differentiate y w.r.t. x

$$\frac{dy}{dx} = 3\frac{d}{dx}(\cos^{-1}x) = 3\left(\frac{-1}{\sqrt{1-x^2}}\right)$$

$$\therefore\ \boxed{\frac{dy}{dx} = \frac{-3}{\sqrt{1-x^2}}}$$

* Derivative of Implicit Functions :-

Implicit functions are expressed in the form $f(x,y)=0$. For implicit functions, we cannot express y in terms of x alone or vice versa. To obtain derivative of such functions, we have following working rule.

Working Rule :-

1) Differentiate each term of the relation w.r. to x using rules of differentiation.
2) Multiply the derivative of y terms by $\frac{dy}{dx}$ to show that y is a function of x.
3) Collect terms with $\frac{dy}{dx}$ on L.H.S. and those without $\frac{dy}{dx}$ on R.H.S. Hence, find $\frac{dy}{dx}$.

Note :- $\frac{d}{dx}[f(y)] = f'(y)\cdot\frac{dy}{dx}$

* Examples :-

Ex-1) If $x^2+y^2=xy$ find $\frac{dy}{dx}$ OR If $x^2+y^2-xy=0$ find $\frac{dy}{dx}$.

→ Given: $x^2+y^2=xy$

Differentiate w.r. to x.

$$\frac{d}{dx}(x^2)+\frac{d}{dx}(y^2)=\frac{d}{dx}(xy)$$

$$\therefore 2x+2y\frac{dy}{dx}=x\frac{dy}{dx}+y(1)$$

$\left[\because \text{Using Product Rule } \frac{d}{dx}(uv)=u\frac{dv}{dx}+v\frac{du}{dx}\right]$

Collect all term containing $\frac{dy}{dx}$ on L.H.S. and other terms on R.H.S.

$$\therefore 2y\frac{dy}{dx}-x\frac{dy}{dx}=y-2x$$

$$\therefore \frac{dy}{dx}(2y-x)=y-2x$$

$$\therefore \boxed{\frac{dy}{dx}=\frac{y-2x}{2y-x}}$$

Ex-2) Find $\frac{dy}{dx}$ at $(2,-1)$ if $x^2+y^2=4xy$

$\rightarrow$ Given: $x^2+y^2=4xy$

Differentiate w.r.t. x

$\therefore \frac{d}{dx}(x^2)+\frac{d}{dx}(y^2)=4\frac{d}{dx}(xy)$ $\left[\because \text{Product Rule} \quad \frac{d}{dx}(u.v)=u\frac{dv}{dx}+v\frac{du}{dx}\right]$

$\therefore 2x+2y\frac{dy}{dx}=4\left[x\frac{dy}{dx}+y(1)\right]$

$\therefore 2x+2y\frac{dy}{dx}=4x\frac{dy}{dx}+4y$

$\therefore$ collecting terms in $\frac{dy}{dx}$ on L.H.S.

$\therefore 2y\frac{dy}{dx}-4x\frac{dy}{dx}=4y-2x$

$\therefore \frac{dy}{dx}(2y-4x)=4y-2x$

$\therefore \frac{dy}{dx}=\frac{4y-2x}{2y-4x}$

$\frac{dy}{dx}=\frac{2(2y-x)}{2(y-2x)}$

$\therefore \frac{dy}{dx}=\frac{2y-x}{y-2x}$

at $(2,-1)$

$\frac{dy}{dx}=\frac{2(-1)-2}{-1-2(2)}=\frac{-2-2}{-1-4}=\frac{-4}{-5}$

$\therefore \boxed{\frac{dy}{dx}=\frac{4}{5}}$

Ex-3) If $x^2+y^2+xy-y=0$, find $\frac{dy}{dx}$ at $(1,2)$.

$\rightarrow$ Given: $x^2+y^2+xy-y=0$

Differentiate w.r.t. x.

$\frac{d}{dx}(x^2)+\frac{d}{dx}(y^2)+\frac{d}{dx}(xy)-\frac{d}{dx}(y)=0$

$\therefore 2x+2y\frac{dy}{dx}+\left[x\frac{dy}{dx}+y(1)\right]-\frac{dy}{dx}=0$

$\therefore 2x+2y\frac{dy}{dx}+x\frac{dy}{dx}+y-\frac{dy}{dx}=0$

$$\therefore 2x\frac{dy}{dx} + x\frac{dy}{dx} - \frac{dy}{dx} = -2x - y$$

$$\therefore \frac{dy}{dx}(2y + x - 1) = -(2x + y)$$

$$\therefore \frac{dy}{dx} = \frac{-(2x+y)}{(2y+x-1)}$$

$$\therefore \left[\frac{dy}{dx}\right]_{at\,(1,2)} = \frac{-[2(1)+2]}{2(2)+1-1} = \frac{-4}{4} = -1$$

$$\therefore \boxed{\left[\frac{dy}{dx}\right]_{at\,(1,2)} = -1}$$

Ex-4) Differentiate with respect to x : $x\log y + y\log x = 0$.

OR

Find $\frac{dy}{dx}$ if $x\log y + y\log x = 0$.

$\longrightarrow$ Given : $x\log y + y\log x = 0$

Differentiate w.r.t. x

$$\therefore \frac{d}{dx}(x\log y) + \frac{d}{dx}(y\log x) = 0 \quad \left[\because \frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}\right]$$

$$\left[x\frac{d}{dx}(\log y) + \log y\frac{d}{dx}(x)\right] + \left[y\frac{d}{dx}(\log x) + \log x\frac{d}{dx}(y)\right] = 0$$

$$\left[x\cdot\frac{1}{y}\frac{dy}{dx} + \log y\right] + \left[y\cdot\frac{1}{x} + \log x\frac{dy}{dx}\right] = 0$$

$$\frac{x}{y}\frac{dy}{dx} + \log y + \frac{y}{x} + \log x\frac{dy}{dx} = 0$$

Collect all terms containing $\frac{dy}{dx}$ on L.H.S. and other terms on R.H.S.

$$\frac{x}{y}\frac{dy}{dx} + \log x\frac{dy}{dx} = -\frac{y}{x} - \log y$$

$$\therefore \frac{dy}{dx}\left[\frac{x}{y} + \log x\right] = -\frac{y}{x} - \log y$$

$$\frac{dy}{dx}\left[\frac{x + y\log x}{y}\right] = \left[\frac{-y - x\log y}{x}\right]$$

$$\therefore \frac{dy}{dx} = \frac{-(y + x\log y)}{x} \times \frac{y}{x + y\log x}$$

$$\therefore \boxed{\frac{dy}{dx} = \frac{-y(y + x\log y)}{x(x + y\log x)}}$$

Ex-5) If $e^x+e^y=e^{x+y}$, find $\frac{dy}{dx}$.

$\longrightarrow$ Given: $e^x+e^y=e^{x+y}$

Differentiate w.r.t x

$$\frac{d}{dx}(e^x)+\frac{d}{dx}(e^y)=\frac{d}{dx}(e^{x+y})$$

$$\therefore e^x+e^y\frac{dy}{dx}=e^{x+y}\frac{d}{dx}(x+y)$$

$$\therefore e^x+e^y\frac{dy}{dx}=e^{x+y}\left[1+\frac{dy}{dx}\right]$$

$$\therefore e^x+e^y\frac{dy}{dx}=e^{x+y}+e^{x+y}\frac{dy}{dx}$$

collect all term containing $\frac{dy}{dx}$ on L.H.S and other terms on R.H.S.

$$\therefore e^y\frac{dy}{dx}-e^{x+y}\frac{dy}{dx}=e^{x+y}-e^x$$

$$\therefore \frac{dy}{dx}(e^y-e^{x+y})=e^{x+y}-e^x$$

$$\therefore \boxed{\frac{dy}{dx}=\frac{e^{x+y}-e^x}{e^y-e^{x+y}}}$$

Ex-6) Find $\frac{dy}{dx}$, if $\cos(x^2+y^2)=\log(xy)$.

$\longrightarrow$ Given: $\cos(x^2+y^2)=\log(xy)$

Differentiate w.r.t. x

$$\frac{d}{dx}\cos(x^2+y^2)=\frac{d}{dx}\log(xy)$$

$$-\sin(x^2+y^2)\frac{d}{dx}(x^2+y^2)=\frac{1}{xy}\frac{d}{dx}(x.y)$$

$$\therefore -\sin(x^2+y^2)\left(2x+2y\frac{dy}{dx}\right)=\frac{1}{xy}\left[x\frac{dy}{dx}+y(1)\right]$$

$$\therefore -2x\sin(x^2+y^2)-2y\sin(x^2+y^2)\frac{dy}{dx}=\frac{1}{y}\frac{dy}{dx}+\frac{1}{x}$$

$$\therefore -2y\sin(x^2+y^2)\frac{dy}{dx}-\frac{1}{y}\frac{dy}{dx}=\frac{1}{x}+2x\sin(x^2+y^2)$$

$$\therefore \frac{dy}{dx}\left[-2y\sin(x^2+y^2)-\frac{1}{y}\right]=\frac{1}{x}+2x\sin(x^2+y^2)$$

$$\therefore \boxed{\frac{dy}{dx}=\frac{\frac{1}{x}+2x\sin(x^2+y^2)}{-2y\sin(x^2+y^2)-\frac{1}{y}}}$$

* Logarithmic Differentiation :-

If a function is expressed as:

1) Product of two or more than two functions.
2) Quotient of functions.
3) In the form $[f(x)]^{g(x)}$

In such types, we have to take logarithms first and then differentiate. This method of differentiation is called Logarithmic differentiation.

For this use following laws of logarithms :

1) $\log(m.n) = \log m + \log n$
2) $\log\left(\frac{m}{n}\right) = \log m - \log n$
3) $\log(m)^n = n.\log m$

* Examples :-

Ex-1) Find $\frac{dy}{dx}$ if $y = x^x$.

→ Given : $y = x^x$

Taking Logarithm on both sides

$\therefore \log y = \log(x^x)$ $\quad [\because \log(m)^n = n \log m]$

$\therefore \log y = x \log x$

Differentiate w.r.t. x

$\therefore \frac{d}{dx}(\log y) = \frac{d}{dx}(x \log x)$ $\quad \left[\because \frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}\right]$

$\therefore \frac{1}{y}\frac{dy}{dx} = x\frac{d}{dx}(\log x) + \log x \frac{d}{dx}(x)$

$\therefore \frac{1}{y}\frac{dy}{dx} = x \times \frac{1}{x} + \log x \,(1)$

$\frac{1}{y}\frac{dy}{dx} = 1 + \log x$

$\therefore \frac{dy}{dx} = y(1 + \log x)$

Substitute value of y,

$\therefore \boxed{\frac{dy}{dx} = x^x(1 + \log x)}$

Ex-2) Differentiate w.r.t. x: $(\tan x)^x$.

$\rightarrow$ Let $y = (\tan x)^x$

Taking logarithm on both sides,

$$\log y = \log(\tan x)^x \qquad [\because \log(m)^n = n \log m]$$

$$\log y = x \log(\tan x)$$

Differentiate w.r.t. x.

$$\therefore \frac{d}{dx}(\log y) = \frac{d}{dx}[x . \log(\tan x)] \qquad \left[\because \frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{dy}{dx}\right]$$

$$\frac{1}{y}\frac{dy}{dx} = x\frac{d}{dx}\log(\tan x) + \log(\tan x)\frac{d}{dx}(x)$$

$$\frac{1}{y}\frac{dy}{dx} = x \times \frac{1}{\tan x} \cdot \sec^2 x + \log(\tan x)\,(1)$$

$$\therefore \frac{dy}{dx} = y\left[\frac{x \sec^2 x}{\tan x} + \log(\tan x)\right]$$

Substitute value of y

$$\therefore \boxed{\frac{dy}{dx} = (\tan x)^x\left[\frac{x \sec^2 y}{\tan x} + \log(\tan x)\right]}$$

Ex-3) Differentiate w.r.t. x: $n^x + x^n + x^x + n^n$.

$\rightarrow$ Let $y = n^x + x^n + x^x + n^n$ —— ①

Put $u = x^x$

Taking logarithm on both sides

$$\therefore \log u = \log(x^x)$$

$$\log u = x \log x \qquad [\because \log(m^n) = n \log m]$$

Differentiate w.r.t. x

$$\therefore \frac{d}{dx}(\log u) = \frac{d}{dx}(x . \log x)$$

$$\frac{1}{u}\frac{du}{dx} = x\frac{d}{dx}(\log x) + \log(x)\frac{d}{dx}(x)$$

$$\frac{1}{u}\frac{du}{dx} = x \times \frac{1}{x} + \log x \times (1)$$

$$\therefore \frac{du}{dx} = u[1 + \log x]$$

Substitute value of u

$$\therefore \frac{d(x^x)}{dx} = x^x(1 + \log x) \text{ —— ②}$$

eq^n ① becomes

$$y = n^x + x^n + x^x + n^n$$

→ Differentiate y w.r.t. x

$$\therefore \frac{dy}{dx} = \frac{d}{dx}(n^x) + \frac{d}{dx}(x^n) + \frac{d}{dx}(x^x) + \frac{d}{dx}(n^n)$$

$$\therefore \frac{dy}{dx} = n^x \log n + n x^{n-1} + \frac{d}{dx}(x^x) + 0$$

$$\left[\because \frac{d}{dx}(a^x) = a^x \log a,\ \frac{d}{dx}(x^n) = n x^{n-1},\ \frac{d}{dx}(k) = 0\right]$$

$$\therefore \boxed{\frac{dy}{dx} = n^x \log n + n x^{n-1} + x^x(1+\log x)}$$ — [from eq^n ②]

Ex-4) If $x^y = e^{x-y}$ show that $\frac{dy}{dx} = \frac{\log x}{(1+\log x)^2}$

→ Given: $x^y = e^{x-y}$

Taking logarithm on both sides

$$\log(x^y) = \log(e^{x-y}) \qquad [\because \log(m^n) = n \log m]$$

$$\therefore y \log x = (x-y)\log_e e \qquad [\because \log_a a = 1]$$

$$\therefore y \log x = x - y$$

$$\therefore y \log x + y = x$$

$$\therefore y(1+\log x) = x$$

$$\therefore y = \frac{x}{\log x + 1}$$

Differentiate y w.r.t. x

$$\therefore \frac{dy}{dx} = \frac{d}{dx}\left[\frac{x}{\log x + 1}\right]$$

$$\left[\text{Quotient Rule } \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}\right]$$

$$\therefore \frac{dy}{dx} = \frac{(\log x + 1)\frac{d}{dx}(x) - x\frac{d}{dx}(\log x + 1)}{(\log x + 1)^2}$$

$$= \frac{(\log x + 1)(1) - x\left(\frac{1}{x} + 0\right)}{(\log x + 1)^2}$$

$$= \frac{(\log x + 1) - x \times \frac{1}{x}}{(\log x + 1)^2}$$

$$\therefore \frac{dy}{dx} = \frac{\log x + 1 - 1}{(1+\log x)^2}$$

$$\therefore \boxed{\frac{dy}{dx} = \frac{\log x}{(1+\log x)^2}}$$

Ex-5) If $x^p \cdot y^q = (x+y)^{p+q}$ show that $\frac{dy}{dx} = \frac{y}{x}$.

$\longrightarrow$ Given: $x^p \cdot y^q = (x+y)^{p+q}$

Taking Logarithm on both sides

$$\log(x^p \cdot y^q) = \log(x+y)^{p+q}$$

$$\left[\because \log(mn) = \log m + \log n \quad , \quad \log(m^n) = n \log m\right]$$

$$\therefore \log(x^p) + \log(y^q) = (p+q)\log(x+y)$$

$$p \cdot \log x + q \log y = (p+q)\log(x+y)$$

Differentiate w. r. t. x

$$p \frac{d}{dx}(\log x) + q \frac{d}{dx}(\log y) = (p+q)\frac{d}{dx}\log(x+y)$$

$$p \cdot \frac{1}{x} + q \cdot \frac{1}{y}\frac{dy}{dx} = (p+q)\frac{1}{x+y}\frac{d}{dx}(x+y)$$

$$\frac{p}{x} + \frac{q}{y}\frac{dy}{dx} = \frac{p+q}{x+y}\left[1 + \frac{dy}{dx}\right]$$

$$\frac{p}{x} + \frac{q}{y}\frac{dy}{dx} = \frac{p+q}{x+y} + \frac{p+q}{x+y}\frac{dy}{dx}$$

Collect all term containing $\frac{dy}{dx}$ on L.H.S. and other terms on R.H.S.

$$\frac{q}{y}\frac{dy}{dx} - \frac{p+q}{x+y}\frac{dy}{dx} = \frac{p+q}{x+y} - \frac{p}{x}$$

$$\therefore \frac{dy}{dx}\left[\frac{q}{y} - \frac{p+q}{x+y}\right] = \frac{x(p+q) - p(x+y)}{x(x+y)}$$

$$\frac{dy}{dx}\left[\frac{q(x+y) - y(p+q)}{y(x+y)}\right] = \frac{px + qx - px - py}{x(x+y)}$$

$$\frac{dy}{dx}\left[\frac{qx + qy - py - qy}{y(x+y)}\right] = \frac{qx - py}{x(x+y)}$$

$$\therefore \frac{dy}{dx}\left[\frac{qx - py}{y(x+y)}\right] = \frac{qx - py}{x(x+y)}$$

$$\therefore \frac{dy}{dx} = \frac{qx - py}{x(x+y)} \times \frac{y(x+y)}{qx - py} = \frac{y}{x}$$

$$\therefore \boxed{\frac{dy}{dx} = \frac{y}{x}}$$

* Derivative of Parametric Functions :-

When the variables x and y are expressed in terms of some other variable (called Parameter) t or θ, the function are called Parametric functions.

If $x = f(t)$ and $y = g(t)$, where t is a parameter then,

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} \quad \text{where } \frac{dx}{dt} \neq 0$$

Ex-1) Find $\frac{dy}{dx}$ if $x = r\cos\theta$, $y = r\sin\theta$

→ Given: $x = r\cos\theta$ $\qquad y = r\sin\theta$.

Differentiate x & y w.r.t. θ

$$\frac{dx}{d\theta} = r\frac{d}{d\theta}\cos\theta$$

$$= r(-\sin\theta)$$

$$\therefore \frac{dx}{d\theta} = -r\sin\theta$$

$$\frac{dy}{d\theta} = r\frac{d}{d\theta}(\sin\theta).$$

$$\therefore \frac{dy}{d\theta} = r\cos\theta$$

By Derivative of Parametric function

$$\therefore \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{r\cos\theta}{-r\sin\theta} = -\frac{\cos\theta}{\sin\theta}$$

$$\therefore \boxed{\frac{dy}{dx} = -\cot\theta}$$

Ex-2) If $x = a\sec t$, $y = b\tan t$ then find $\frac{dy}{dx}$ at $t = \frac{\pi}{2}$.

→ Given: $x = a\sec t$ $\qquad y = b\tan t$

Differentiate x & y w.r.t. t

$$\therefore \frac{dx}{dt} = a\frac{d}{dt}(\sec t)$$

$$\frac{dx}{dt} = a\sec t \cdot \tan t$$

$$\frac{dy}{dt} = b\frac{d}{dt}(\tan t)$$

$$\frac{dy}{dt} = b \cdot \sec^2 t.$$

By Derivative of Parametric Function

$$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{b \cdot \sec^2 t}{a\sec t \cdot \tan t} = \frac{b \cdot \sec t \cdot \sec t}{a \cdot \sec t \cdot \tan t}$$

$$\therefore \frac{dy}{dx} = \frac{b}{a}\frac{\sec t}{\tan t} = \frac{b}{a} \cdot \frac{\left(\frac{1}{\cos t}\right)}{\left(\frac{\sin t}{\cos t}\right)} = \frac{b}{a}\frac{1}{\sin t}$$

$\therefore \dfrac{dy}{dx} = \dfrac{b}{a}\operatorname{cosec} t$

at $t = \dfrac{\pi}{2}$

$\therefore \left[\dfrac{dy}{dx}\right]_{at\ t=\frac{\pi}{2}} = \dfrac{b}{a}\operatorname{cosec}\left(\dfrac{\pi}{2}\right) = \dfrac{b}{a}(1) = \dfrac{b}{a}$

$\therefore \boxed{\dfrac{dy}{dx} = \dfrac{b}{a}}$

Ex-3) If $x = a(1+\cos\theta)$, $y = a(1-\cos\theta)$ find $\dfrac{dy}{dx}$

$\longrightarrow$ Given: $x = a(1+\cos\theta)$, $y = a(1-\cos\theta)$

Differentiate x & y w.r.t. θ

$\therefore \dfrac{dx}{d\theta} = a\dfrac{d}{d\theta}(1+\cos\theta)$ & $\dfrac{dy}{d\theta} = a\dfrac{d}{d\theta}(1-\cos\theta)$

$= a(0-\sin\theta)$ $\qquad = a[0-(-\sin\theta)]$

$\therefore \dfrac{dx}{d\theta} = -a\sin\theta$ $\qquad \dfrac{dy}{d\theta} = a\sin\theta$

By the Derivative of Parametric function

$\therefore \dfrac{dy}{dx} = \dfrac{\left(\dfrac{dy}{d\theta}\right)}{\left(\dfrac{dx}{d\theta}\right)} = \dfrac{a\sin\theta}{-a\sin\theta} = -1$

$\therefore \boxed{\dfrac{dy}{dx} = -1}$

Ex-4) If $x = a(2\theta - \sin 2\theta)$ and $y = a(1-\cos 2\theta)$, find $\dfrac{dy}{dx}$ at $\theta = \dfrac{\pi}{4}$.

$\longrightarrow$ Given: $x = a(2\theta - \sin 2\theta)$ & $y = a(1-\cos 2\theta)$

Differentiate x & y w.r.t. θ

$\dfrac{dx}{d\theta} = a\dfrac{d}{d\theta}(2\theta - \sin 2\theta)$

$= a[2(1) - \cos(2\theta)\cdot 2]$

$= 2a[1-\cos(2\theta)]$

$\dfrac{dx}{d\theta} = 2a(2\sin^2\theta)$

$\dfrac{dx}{d\theta} = 4a\sin^2\theta$

$\dfrac{dy}{d\theta} = a\dfrac{d}{d\theta}(1-\cos 2\theta)$

$\dfrac{dy}{d\theta} = a[0-(-\sin 2\theta)\cdot 2]$

$= 2a\cdot\sin(2\theta)$

$\dfrac{dy}{d\theta} = 2a\cdot 2\sin\theta\cos\theta$

$\dfrac{dy}{d\theta} = 4a\sin\theta\cdot\cos\theta$

By the derivative of Parametric function

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{4a.\sin\theta.\cos\theta}{4a\sin^2\theta} = \frac{4a.\sin\theta.\cos\theta}{4a.\sin\theta.\sin\theta}$$

$$\therefore \frac{dy}{dx} = \frac{\cos\theta}{\sin\theta}$$

$$\therefore \boxed{\frac{dy}{dx} = \cot\theta}$$

$$\left[\frac{dy}{dx}\right]_{at\ \theta=\frac{\pi}{4}} = \cot\left(\frac{\pi}{4}\right) = \cot(45^\circ) = 1$$

Ex-5) Find $\frac{dy}{dx}$ if $x = \frac{1}{t}$ and $y = 1 - \frac{1}{t}$

→ Given: $x = \frac{1}{t}$ and $y = 1 - \frac{1}{t}$

Differentiate x & y w.r. to t

$$\therefore \frac{dx}{dt} = -\frac{1}{t^2} \quad \& \quad \frac{dy}{dt} = \frac{d}{dt}(1) - \frac{d}{dt}\left(\frac{1}{t}\right)$$

$$\frac{dy}{dt} = 0 - \left(-\frac{1}{t^2}\right)$$

$$\frac{dy}{dt} = \frac{1}{t^2}$$

By the Derivative of Parametric function

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{d}\right)}{\left(\frac{dx}{d}\right)} = \frac{\left(\frac{1}{t^2}\right)}{\left(-\frac{1}{t^2}\right)} = -1$$

$$\therefore \boxed{\frac{dy}{dx} = -1}$$

* Second Order Differentiation :-

We have denoted the first derivative of $y=f(x)$ by $f'(x)$ or $\frac{dy}{dx}$. This derivative called the first derivative of the function f and it is function of x. This first derivative can again be differentiated with respect to x and the result is called the second derivative of the function f. It is denoted by $f''(x)$ or $\frac{d^2y}{dx^2}$. Thus $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$

For example:

Let $y = x^5 + 4$

Differentiate y w.r.t. x

$\frac{dy}{dx} = 5x^4$ —— first order derivative

Again differentiate w.r.t. x

$\frac{d}{dx}\left(\frac{dy}{dx}\right) = 5\frac{d}{dx}(x^4)$

$\frac{d^2y}{dx^2} = 20x^3$ —— Second Order derivative

Again differentiate w.r.t. x

$\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = 20\frac{d}{dx}(x^3)$

$\frac{d^3y}{dx^3} = 60x^2$ —— Third order derivative

$\vdots$

$\frac{d^ny}{dx^n} =$ —— n^{th} order derivative

Note:- The n^{th} derivative of a function $y=f(x)$ is denoted by $f^n(x)$ or $\frac{d^ny}{dx^n}$ or $y^{(n)}$ or y_n.

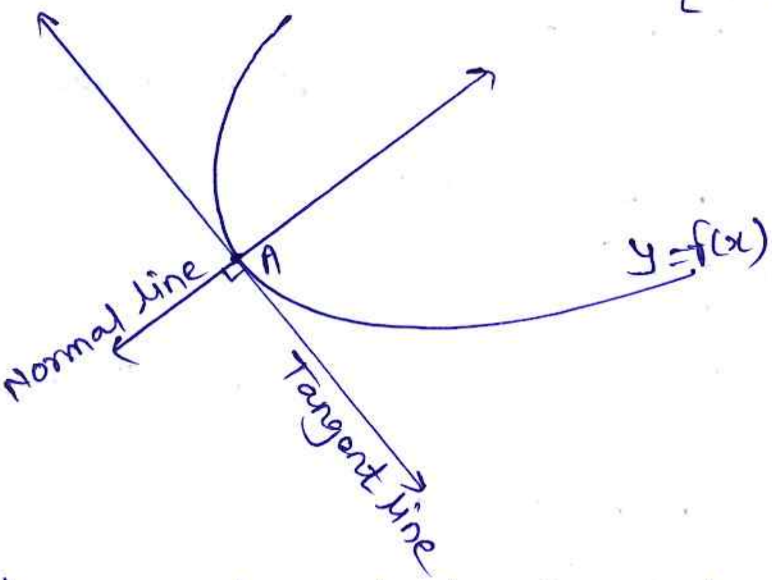
* Application of Derivative :-

* Tangent and Normal :-

In this section, we shall use differentiation to find the equation of the tangent line and the normal line to a curve at a given point.

Recall that the equation of a straight line passing through a given point (x_1, y_1) having finite slope 'm' is given by $y - y_1 = m(x - x_1)$

Note that the slope of the tangent to the curve $y=f(x)$ at the point (x_1, y_1) is given by $\left[\frac{dy}{dx}\right]_{at\,(x_1,y_1)} = m.$



So the equation of the tangent at (x_1, y_1) to the curve $y = f(x)$ is given by $\boxed{y - y_1 = m(x - x_1)}$

Also, since the normal is perpendicular to the tangent, the slope of the normal to the curve $y = f(x)$ at (x_1, y_1) is $-\frac{1}{m}$ if $m \neq 0$.

Therefore, the equation of normal to the curve $y = f(x)$ at (x_1, y_1) is given by, $\boxed{y - y_1 = -\frac{1}{m}(x - x_1)}$

Note:- If a tangent line to the curve $y = f(x)$ make an angle θ with x-axis in the positive direction, then

$\boxed{\frac{dy}{dx} = \text{Slope of tangent} = \tan\theta}$ i.e. $\boxed{m = \tan\theta}$

* Examples :-

Ex-1) Find the gradient of the tangent of the curve $y=\sqrt{x^3}$ at $x=4$

→ Given equation of the curve

$$y=\sqrt{x^3} \qquad \left[\because \sqrt[n]{x^m}=x^{m/n}\right]$$

$$y=x^{3/2}$$

Differentiate w.r.t. x

$$\therefore \frac{dy}{dx}=\frac{d}{dx}\left(x^{3/2}\right)=\frac{3}{2}x^{\frac{3}{2}-1}=\frac{3}{2}x^{\frac{1}{2}}=\frac{3}{2}\sqrt{x}$$

$$\therefore \frac{dy}{dx}=\frac{3}{2}\sqrt{x}$$

$$\therefore \text{slope of tangent}=\left[\frac{dy}{dx}\right]_{at\ x=4}=\frac{3}{2}\sqrt{4}=\frac{3}{2}\times 2=3$$

Ex-2) Find the point on the curve $y=3x-x^2$ at which slope is -5.

→ Given equation of the curve is,

$$y=3x-x^2 \quad \text{—(1)} \quad \text{and} \quad \text{slope } (m)=-5$$

we know that

Slope of curve $(m)=\dfrac{dy}{dx}$

$$\therefore \frac{dy}{dx}=-5$$

Differentiate y w.r.t. x

$$\frac{dy}{dx}=\frac{d}{dx}(3x)-\frac{d}{dx}(x^2)$$

$$\therefore -5=3-2x$$

$$2x=3+5$$

$$\therefore 2x=8$$

$$\therefore x=\frac{8}{2}$$

$$\therefore \boxed{x=4}$$

Put $x=4$ in equation (1)

$$y=3x-x^2$$

$$y=3(4)-(4)^2$$

$$y=12-16$$

$$\therefore y=-4$$

$$\boxed{\text{Required point}=(4,-4)}$$

Ex-3) Find the point on the curve $y = x^2 - 6x + 8$ where the tangent is parallel to x-axis.

→ The given equation of curve

$$y = x^2 - 6x + 8 \quad \text{——} \quad (1)$$

Differentiate y w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dx}(x^2) - 6\frac{d}{dx}(x) + \frac{d}{dx}(8)$$

$$= 2x - 6(1) + 0$$

$$\therefore \frac{dy}{dx} = 2x - 6 \quad \text{——} \quad (2)$$

Now The given tangent is parallel to the X-axis

$$\therefore \frac{dy}{dx} = 0$$

eqn. (2) becomes

$$\therefore 0 = 2x - 6$$

$$\therefore 2x = 6$$

$$\therefore x = \frac{6}{2}$$

$$\therefore \boxed{x = 3}$$

Put $x = 3$ in equation (1)

$$y = (3)^2 - 6(3) + 8$$

$$y = 9 - 18 + 8$$

$$y = -1$$

$$\boxed{\text{Required point} = (3, -1)}$$

Ex-4) Find the inclination of the tangent to the curve $y = e^{2x}$ at $(1, -3)$

→ The given equation of curve

$$y = e^{2x}$$

Differentiate y w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dx}(e^{2x}) = e^{2x}(2) = 2 \cdot e^{2x}$$

$$\left[\frac{dy}{dx}\right]_{at\,(1,-3)} = 2 \cdot e^{2(1)} = 2 \cdot e^2 = \text{slope of tangent}$$

Let θ is the inclination of the tangent.

$\tan\theta$ = slope of tangent

$\tan\theta = 2 \cdot e^2$

$\therefore \boxed{\theta = \tan^{-1}(2 \cdot e^2)}$

Ex-5) Find the slope of tangent and normal to the curve $y = x^2 - 2x + 1$ at $P(0,1)$

→ Given equation of the curve

$$y = x^2 - 2x + 1$$

Differentiate y w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}(x^2) - 2\frac{d}{dx}(x) + \frac{d}{dx}(1)$$

$$= 2x - 2(1) + 0$$

$$\therefore \frac{dy}{dx} = 2x - 2$$

$\therefore$ Slope of tangent $= \left[\frac{dy}{dx}\right]_{at\,(0,1)} = 2(0) - 2 = -2$

$\therefore$ $\boxed{\text{Slope of tangent} = m = -2}$

Now, slope of normal $= \dfrac{-1}{\text{Slope of tangent}} = \dfrac{-1}{-2} = \dfrac{1}{2}$

$\therefore$ $\boxed{\text{Slope of normal} = \frac{1}{2}}$

* **Equation of Tangent and Normal :-**

Equation of tangent is given by

$$\boxed{y - y_1 = m(x - x_1)}$$

Equation of normal is given by

$$\boxed{y - y_1 = \frac{-1}{m}(x - x_1)}$$

* Examples :-

Ex-1) Find equation of tangent and normal to the curve $y = x(2-x)$ at point $(2,0)$.

→ Given equation of the curve

$\therefore \quad y = x(2-x)$

$y = 2x - x^2$

Differentiate y w.r.t. x

$\therefore \dfrac{dy}{dx} = 2\dfrac{d}{dx}(x) - \dfrac{d}{dx}(x^2)$

$= 2(1) - 2x$

$\therefore \dfrac{dy}{dx} = 2 - 2x$

$\therefore$ Slope of tangent $= \left[\dfrac{dy}{dx}\right]_{at\,(2,0)} = 2 - 2(2) = 2 - 4 = -2$

$\therefore \boxed{\text{Slope} = m = -2}$

Now, Equation of tangent at $(2,0)$

$y - y_1 = m(x - x_1)$

$y - 0 = -2(x-2)$

$y = -2x + 4$

$\therefore \boxed{2x + y - 4 = 0}$

Equation of normal at $(2,0)$

$y - y_1 = -\dfrac{1}{m}(x - x_1)$

$y - 0 = \dfrac{-1}{-2}(x-2)$

$y = \dfrac{1}{2}(x-2)$

$\therefore 2y = 1(x-2)$

$\therefore 2y = x - 2$

$\therefore \boxed{x - 2y - 2 = 0}$

Ex-2) Find the equation of tangent and normal to the curve $2x^2 - xy + 3y^2 = 18$ at $(3,1)$

→ Given equation of the curve

$$2x^2 - xy + 3y^2 = 18$$

Differentiate w.r.t. x

$$2\frac{d}{dx}(x^2) - \frac{d}{dx}(xy) + 3\frac{d}{dx}(y^2) = \frac{d}{dx}(18)$$

$$2(2x) - \left[x\frac{dy}{dx} + y(1)\right] + 3(2y)\frac{dy}{dx} = 0$$

$$4x - x\frac{dy}{dx} - y + 6y\frac{dy}{dx} = 0$$

$$\therefore \quad 6y\frac{dy}{dx} - x\frac{dy}{dx} = y - 4x$$

$$\therefore \quad \frac{dy}{dx}(6y - x) = y - 4x$$

$$\therefore \quad \frac{dy}{dx} = \frac{y-4x}{6y-x}$$

$$\left[\frac{dy}{dx}\right]_{at(3,1)} = \frac{1-4(3)}{6(1)-3} = \frac{-11}{3}$$

$$\therefore \quad \boxed{\text{slope} = m = \frac{-11}{3}}$$

Now Equation of tangent at $(3,1)$

$$y - y_1 = m(x - x_1)$$

$$y - 1 = \frac{-11}{3}(x-3)$$

$$\therefore \quad 3(y-1) = -11(x-3)$$

$$\therefore \quad 3y - 3 = -11x + 33$$

$$\therefore \quad 11x + 3y - 3 - 33 = 0$$

$$\therefore \quad \boxed{11x + 3y - 36 = 0}$$

Equation of normal at $(3,1)$

$$y - y_1 = \frac{-1}{m}(x - x_1)$$

$$\therefore \quad y - 1 = \frac{-1}{\left(\frac{-11}{3}\right)}(x-3)$$

$$\therefore \quad y - 1 = \frac{3}{11}(x-3)$$

$$\therefore \quad 11(y-1) = 3x - 9$$

$$\therefore \quad 11y - 11 = 3x - 9$$

$\therefore 3x - 11y + 11 - 9 = 0$

$\therefore \boxed{3x - 11y + 2 = 0}$

Ex-3) Find equation of tangent to the circle $x^2+y^2+6x-6y-7=0$ at a point it cuts the x-axis.

$\longrightarrow$ Given equation of curve is

$$x^2+y^2+6x-6y-7=0 \quad \text{——— (1)}$$

It cuts the x-axis $\therefore y = 0$

$\therefore x^2+0^2+6x-6(0)-7=0$

$\therefore x^2+6x-7=0$

$\therefore (x+7)(x-1)=0$

$\therefore x+7=0 \quad \text{or} \quad x-1=0$

$\therefore x=-7 \quad \text{or} \quad x=1$

curve cuts the x-axis at two points $(-7,0)$ & $(1,0)$

Now, Differentiate (1) w.r.t. x

$$\frac{d}{dx}(x^2)+\frac{d}{dx}(y^2)+6\frac{d}{dx}(x)-6\frac{d}{dx}(y)-\frac{d}{dx}(7)=0.$$

$\therefore 2x+2y\frac{dy}{dx}+6-6\frac{dy}{dx}=0$

$\therefore 2y\frac{dy}{dx}-6\frac{dy}{dx}=-2x-6$

$\therefore \frac{dy}{dx}(2y-6)=-2x-6$

$\therefore \frac{dy}{dx}=\frac{-2x-6}{2y-6}$

$\therefore \left[\frac{dy}{dx}\right]_{at\ (-7,0)}=\frac{-2(-7)-6}{2(0)-6}=\frac{8}{-6}=-\frac{4}{3}$

$\therefore \boxed{\text{Slope} = m = -\frac{4}{3}}$

Equation of tangent at $(-7,0)$

$y-y_1=m(x-x_1)$

$y-0=-\frac{4}{3}[x-(-7)]$

$3y=-4(x+7)$

$3y=-4x-28$

$\therefore \boxed{4x+3y+28=0}$

$$\therefore \left[\frac{dy}{dx}\right]_{at(1,0)} = \frac{-2(1)-6}{2(0)-6} = \frac{-8}{-6} = \frac{4}{3}$$

$$\therefore \boxed{\text{slope} = m = \frac{4}{3}}$$

Equation of tangent at $(1,0)$

$$y - y_1 = m(x - x_1)$$

$$y - 0 = \frac{4}{3}(x-1)$$

$$\therefore 3y = 4x - 4$$

$$\therefore \boxed{4x - 3y - 4 = 0}$$

Ex-4) The equation of the tangent at the point $(2,3)$ on the curve $y = ax^3 + b$ is $y = 4x - 5$, find the value of $a+b$.

→ Given equation of curve

$$y = ax^2 + b \quad \text{——— (1)}$$

Differentiate y w.r.t. x

$$\therefore \frac{dy}{dx} = a\frac{d}{dx}(x^3) + \frac{d}{dx}(b) = a(3x^2) + 0$$

$$\frac{dy}{dx} = 3ax^2$$

$$\therefore \left[\frac{dy}{dx}\right]_{at(2,3)} = 3a(2)^2 = 12a \quad \text{——— (2)}$$

Given equation of tangent

$$y = 4x - 5$$

Differentiate w.r.t. x

$$\frac{dy}{dx} = 4\frac{d}{dx}(x) - \frac{d}{dx}(5) = 4(1) - 0$$

$$\therefore \frac{dy}{dx} = 4 = \text{slope of tangent} \quad \text{——— (3)}$$

From (2) & (3)

$$12a = 4$$

$$a = \frac{4}{12} = \frac{1}{3}$$

$$\therefore \boxed{a = \frac{1}{3}}$$

Put $a = \frac{1}{3}$ in equation (1), we get

$$y = ax^3 + b \quad \text{where } x = 2, y = 3$$

$$3 = \frac{1}{3}(2)^3 + b$$

$$3 = \frac{8}{3} + b$$

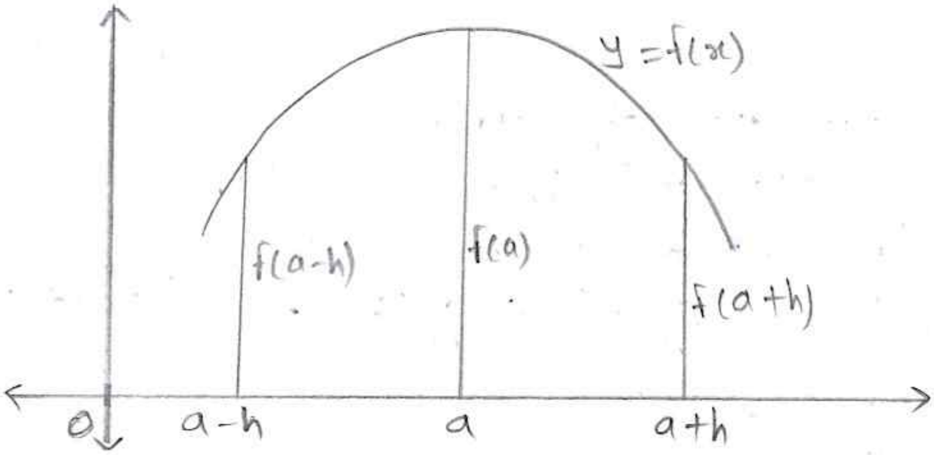
$$\therefore b = 3 - \frac{8}{3} = \frac{9-8}{3} = \frac{1}{3}$$

$$\therefore \boxed{b = \frac{1}{3}}$$

* Maxima and Minima :-

Suppose a function $f(x)$ is defined over the interval $(a-h, a+h)$, where h is a small positive quantity.

Maximum value :-



If the value of the function $f(x)$ at $x = a$ is maximum in a small interval $(a-h, a+h)$, then we say that $f(x)$ is maximum at $x = a$. Thus, if $f(a) > f(a-h)$ and $f(a) > f(a+h)$, where h is a small positive number, then we say that $f(x)$ is maximum at $x = a$.

The following two conditions must be satisfied for any function $f(x)$ to be maximum at $x = a$.

i) $f'(a) = 0$ (Necessary condition)

ii) $f''(a) < 0$ (Sufficient condition)

Minimum Value :-

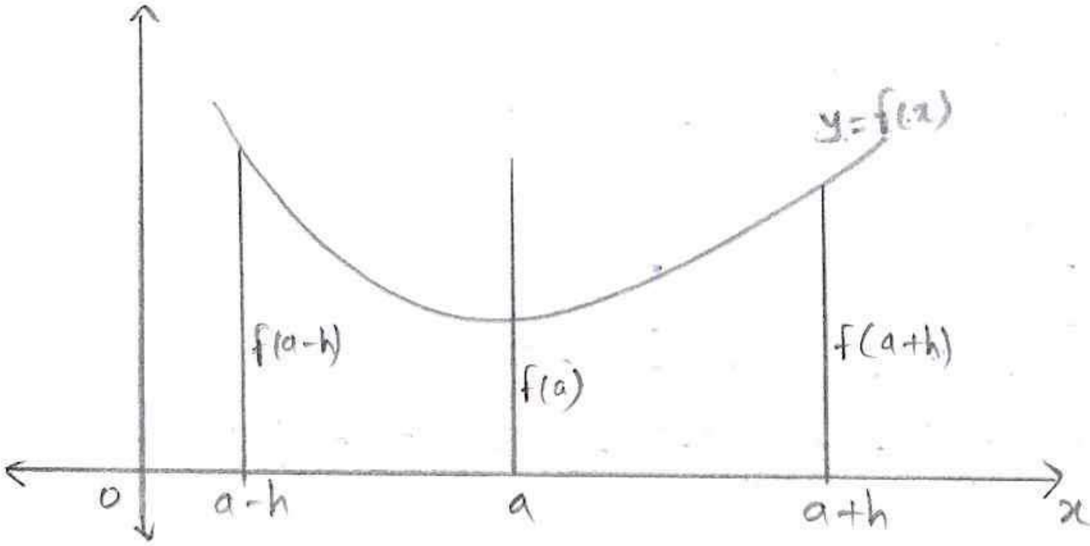
If the value of the function $f(x)$ at $x = a$ is minimum in the small interval $(a-h, a+h)$, then we say that $f(x)$ is minimum at $x = a$.

Thus, if $f(a) < f(a-h)$ and $f(a) < f(a+h)$ where h is a very small positive number, then we say that $f(x)$ is minimum at $x = a$.

The following two conditions must be satisfied for any function $f(x)$ to be minimum at $x = a$.

i) $f'(a) = 0$ (Necessary Condition)

ii) $f''(a) > 0$ (Sufficient condition).



Working rule for finding maxima and minima :-

i) Find $\frac{dy}{dx}$

ii) Put $\frac{dy}{dx} = 0$ and find stationary value, suppose 'a' and 'b' are two stationary values.

iii) Now find $\frac{d^2y}{dx^2}$

iv) Find $\frac{d^2y}{dx^2}$ at $x = a$ and $x = b$

v) If $\frac{d^2y}{dx^2} > 0$ i.e. positive at $x = a$ then $f(x)$ is minimum at $x = a$.

vi) If $\frac{d^2y}{dx^2} < 0$ i.e. negative at $x = b$ then $f(x)$ is maximum at $x = b$.

vii) Find y at $x = a$ and $x = b$ to get minimum and maximum value,

* Examples :-

Ex-1) Find the maximum and minimum value of $x^3 - 9x^2 + 24x$.

→ Let $y = x^3 - 9x^2 + 24x$ ——— ①

Differentiate y w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dx}(x^3) - 9\frac{d}{dx}(x^2) + 24\frac{d}{dx}(x)$$

$$\frac{dy}{dx} = 3x^2 - 9(2x) + 24(1)$$

$$\frac{dy}{dx} = 3x^2 - 10x + 24 \quad \text{——— ②}$$

Consider $\frac{dy}{dx} = 0$

$\therefore 3x^2 - 18x + 24 = 0$

$\therefore x^2 - 6x + 8 = 0$

$\therefore (x-4)(x-2) = 0$

$\therefore x - 4 = 0 \quad$ or $\quad x - 2 = 0$

$\therefore \boxed{x = 4} \quad$ or $\quad \boxed{x = 2}$

Again Differentiate eqn ② w.r.t. x

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = 3\frac{d}{dx}(x^2) - 18\frac{d}{dx}(x) + \frac{d}{dx}(24)$$

$$\therefore \frac{d^2y}{dx^2} = 3(2x) - 18(1) + 0$$

$$\frac{d^2y}{dx^2} = 6x - 18$$

$\left[\frac{d^2y}{dx^2}\right]_{at\, x=4} = 6(4) - 18 = 24 - 18 = 6 > 0$ i.e. (positive)

$\therefore$ y is minimum at $\boxed{x = 4}$

$Y_{min} = (4)^3 - 9(4)^2 + 24(4) = 64 - 144 + 96 = 16$

$\boxed{Y_{min} = 16}$

$\left[\frac{d^2y}{dx^2}\right]_{at\, x=2} = 6(2) - 18 = 12 - 18 = -6 < 0$ i.e (negative)

$\therefore$ y is maximum at $\boxed{x = 2}$

$\therefore Y_{max} = (2)^3 - 9(2)^2 + 24(2) = 8 - 36 + 48 = 20$

$\therefore \boxed{Y_{max} = 20}$

Ex-2) Discuss maxima and minima of the function "$\tan x - 2x$".

$\rightarrow$ Let $y = \tan x - 2x$ ——— ①

Differentiate w.r.t. x

$\therefore \frac{dy}{dx} = \frac{d}{dx}(\tan x) - 2\frac{d}{dx}(x) = \sec^2 x - 2(1)$

$\therefore \frac{dy}{dx} = \sec^2 x - 2$ ——— ②

Consider $\frac{dy}{dx} = 0$

$\therefore \sec^2 x - 2 = 0$

$\therefore 1 + \tan^2 x - 2 = 0$

$\therefore \tan^2 x - 1 = 0$

$\therefore \tan^2 x = 1$

$\therefore \tan x = \pm 1$

$\therefore \tan x = 1$ OR $\tan x = -1$

$\therefore \tan x = \tan\left(\frac{\pi}{4}\right)$ OR $\tan x = -\tan\left(\frac{\pi}{4}\right)$

$\therefore \boxed{x = \frac{\pi}{4}}$ OR $\tan x = \tan\left(-\frac{\pi}{4}\right)$

OR $\boxed{x = -\frac{\pi}{4}}$

Again differentiate equation ② w.r.t. x

$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(\sec^2 x) - \frac{d}{dx}(2)$

$\therefore \frac{d^2y}{dx^2} = 2\sec x \frac{d}{dx}(\sec x) - 0$

$\therefore \frac{d^2y}{dx^2} = 2\sec x \cdot \sec x \cdot \tan x$

$\therefore \frac{d^2y}{dx^2} = 2\sec^2 x \tan x$

$\left[\frac{d^2y}{dx^2}\right]_{at\ x=\frac{\pi}{4}} = 2\sec^2\left(\frac{\pi}{4}\right) \cdot \tan\left(\frac{\pi}{4}\right) = 2(\sqrt{2})^2 \cdot 1 = 2 \cdot 2 \cdot 1 = 4 > 0$

y is minimum at $x = \frac{\pi}{4}$

$y_{min} = \tan\left(\frac{\pi}{4}\right) - 2\left(\frac{\pi}{4}\right) = 1 - \frac{\pi}{2}$

$\therefore \boxed{y_{min} = 1 - \frac{\pi}{2}}$

$\left[\frac{d^2y}{dx^2}\right]_{at\ x=-\frac{\pi}{4}} = 2\sec^2\left(-\frac{\pi}{4}\right) \cdot \tan\left(-\frac{\pi}{4}\right) = 2(\sqrt{2})^2(-1)$

$= 2 \cdot (2)(-1) = -4 < 0$ i.e (Negative)

y is maximum at $x = -\frac{\pi}{4}$

$$\therefore Y_{max} = \tan\left(-\frac{\pi}{4}\right) - 2\left(-\frac{\pi}{4}\right)$$

$$\therefore \boxed{Y_{max} = -1 + \frac{\pi}{2}}$$

Ex-3) Divide 80 into two parts such that their product is maximum.

→ Let one part of 80 be x.

So other part will be $80-x$

Now, Product $p = x(80-x)$

$$p = 80x - x^2 \quad \text{——— (1)}$$

Differentiate w.r.t. x

$$\frac{dp}{dx} = 80 - 2x \quad \text{——— (2)}$$

Now, $$\frac{dp}{dx} = 0$$

$$\therefore 0 = 80 - 2x$$

$$\therefore 2x = 80$$

$$\therefore x = \frac{80}{2}$$

$$\therefore \boxed{x = 40}$$

Again differentiate (2) w.r.t. x

$$\frac{d^2p}{dx^2} = 0 - 2 = -2$$

$$\left(\frac{d^2p}{dx^2}\right)_{at\ x=40} = -2$$

$\therefore$ $\boxed{x = 40 \text{ gives us product maximum}}$

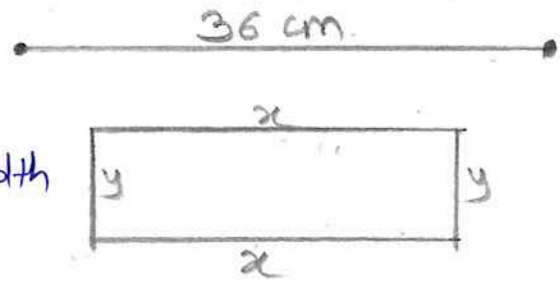
One part $= x = 40$

Other part $= 80 - x = 80 - 40 = 40$

Ex-4) A metal wire 36 cm long is bent to form a rectangle find its dimensions when its area is maximum.

→ Let x = Length of rectangle.

y = Breadth of rectangle.



$\therefore$ Area of rectangle = Length × Breadth

$$\therefore A = x \times y \quad \text{——— (1)}$$

and perimeter = Sum of all sides

$\therefore \quad 36 = x + x + y + y$

$36 = 2x + 2y$

$36 = 2(x + y)$

$\therefore \quad \frac{36}{2} = x + y$

$\therefore \quad x + y = 18$

$y = 18 - x \quad \text{———} \quad (2)$

Putting the value of y in equation (1)

$\therefore \quad A = x.(18 - x)$

$\therefore \quad A = 18x - x^2 \quad \text{———} \quad (3)$

Differentiate w.r.t. x

$$\frac{dA}{dx} = \cancel{18}\frac{d}{dx}(x) - \frac{d}{dx}(x^2)$$

$$\frac{dA}{dx} = 18 - 2x \quad \text{———} \quad (4)$$

But $\frac{dA}{dx} = 0$ for maxima or minima

$\therefore \quad 0 = 18 - 2x$

$\therefore \quad 2x = 18$

$\therefore \quad x = \frac{18}{2}$

$\therefore \quad \boxed{x = 9}$

Now, Again differentiate eqn (4) w.r.t. x

$$\therefore \quad \frac{d^2A}{dx^2} = \frac{d}{dx}(18) - 2\frac{d}{dx}(x) = 0 - 2(1) = -2 < 0$$

$$\therefore \left(\frac{d^2A}{dx^2}\right)_{at\ x = 9} = -2 < 0$$

$\therefore$ $\boxed{\text{at } x = 9, \text{ A has maximum value}}$

$\therefore$ Length of the rectangle $= x = 9$

$\therefore$ Breadth of the rectangle $= y = 18 - x = 18 - 9 = 9$

$\therefore$ $\boxed{\text{So it is square}}$

Ex-5) A manufacture can sell x items at price of ₹ $(330-x)$ each. The cost of producing x-items in ₹ is $x^2+10x+12$. How many items must be sold so that his profit is maximum.

→ The selling cost of each item $= 330 - x$ ₹

The selling cost of x item $= x(330-x)$

cost of producing x-items $= x^2+10x+12$

$\therefore$ Profit = selling cost − Producing cost

$$\therefore \quad P = x(330-x) - (x^2+10x+12)$$

$$P = 330x - x^2 - x^2 - 10x - 12$$

$$P = -2x^2 + 320x - 12 \quad \text{——— (1)}$$

Differentiate w.r.t. x

$$\therefore \frac{dP}{dx} = -2\frac{d}{dx}(x^2) + 320\frac{d}{dx}(x) - \frac{d}{dx}(12)$$

$$= -2(2x) + 320(1) - 0$$

$$\frac{dP}{dx} = -4x + 320 \quad \text{——— (2)}$$

For maximum Profit put $\frac{dP}{dx} = 0$

$$0 = -4x + 320$$

$$\therefore \quad 4x = 320$$

$$x = \frac{320}{4}$$

$$\therefore \boxed{x = 80}$$

Again differentiate (2) w.r.t. x

$$\frac{d}{dx}\left(\frac{dP}{dx}\right) = -4\frac{d}{dx}(x) + \frac{d}{dx}(320)$$

$$\therefore \frac{d^2P}{dx^2} = -4(1) + 0 = -4 < 0 \text{ (Negative)}$$

$\therefore$ Profit is maximum at $x = 80$

Hence the number of items to be sold to for maximum Profit is 80.

Ex-6) The rate of working of an engine is given by the expression $10v + \frac{4000}{v}$, where v is the speed of the engine. Find the speed at which the rate of working is the least.

→ The rate of working is

$$w = 10V + \frac{4000}{V}$$

Differentiate w.r.t. V

$$\therefore \frac{dw}{dV} = 10 - \frac{4000}{V^2} \quad \text{——— (1)}$$

Put $\frac{dw}{dV} = 0$

$$0 = 10 - \frac{4000}{V^2}$$

$$\frac{4000}{V^2} = 10$$

$$V^2 = 400$$

$$\therefore V = 20, -20$$

Again differentiate (1) w.r.t. V

$$\therefore \frac{d^2w}{dV^2} = \frac{d}{dV}(10) - 4000\frac{d}{dV}\left(\frac{1}{V^2}\right)$$

$$= 0 - 4000\frac{d}{dV}(V^{-2})$$

$$= -4000(-2V^{-3})$$

$$\therefore \frac{d^2w}{dV^2} = \frac{8000}{V^3}$$

$$\left[\frac{d^2w}{dV^2}\right]_{at\ V=20} = \frac{8000}{(20)^3} = \frac{8000}{8000} = 1 > 0$$

The speed is V = 20 at which rate of working is least (minimum)

* Radius of curvature :-

The radius of curvature of a curve defined as the radius of the approximate circle at a particular point. It is the length of the curvature vector. As the curve moves, the radius changes. It is denoted by ρ. To find radius of a curvature at the particular point us the below given formula.

$$\rho\ (\text{Rho}) = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\left|\frac{d^2y}{dx^2}\right|}$$

* Examples :-

Ex-1) Find radius of curvature to the curve $y=x^3$ at $(2,8)$.

$\longrightarrow$ Given $y = x^3$

Differentiate y w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dx}(x^3) = 3x^2$$

$$\therefore \frac{dy}{dx} = 3x^2 \text{ ——— (1)}$$

$$\therefore \left[\frac{dy}{dx}\right]_{at\ (2,8)} = 3(2)^2 = 3\times 4 = 12$$

Again differentiate (1) w.r.t. x

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = 3\frac{d}{dx}(x^2)$$

$$= 3(2x)$$

$$\therefore \frac{d^2y}{dx^2} = 6x$$

$$\therefore \left[\frac{d^2y}{dx^2}\right]_{at\ (2,8)} = 6(2) = 12$$

Radius of curvature

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\left|\frac{d^2y}{dx^2}\right|}$$

$$\rho = \frac{\left[1+(12)^2\right]^{3/2}}{|12|} = \frac{(145)^{3/2}}{12} = 145.50 \text{ unit}$$

$\therefore$ $\boxed{\rho = 145.50 \text{ unit}}$

Ex-2) Find the radius of curvature of the curve $y=e^x$ at the point where it crosses the y-axis.

$\rightarrow$ Given equation of curve

$$y = e^x$$

Differentiate w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dx}(e^x)$$

$$\therefore \quad \frac{dy}{dx} = e^x$$

Again differentiate w.r.t. x

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(e^x)$$

$$\therefore \quad \frac{d^2y}{dx^2} = e^x$$

But, curve crosses y-axis i.e. $x=0$

$$\left(\frac{dy}{dx}\right)_{at\ x=0} = e^0 = 1$$

$$\left(\frac{d^2y}{dx^2}\right)_{at\ x=0} = e^0 = 1$$

Radius of curvature

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\left|\frac{d^2y}{dx^2}\right|} = \frac{\left[1+(1)^2\right]^{3/2}}{|1|} = \frac{(2)^{3/2}}{1} = (2)^{3/2}$$

$\therefore$ $\boxed{\rho = 2.828 \text{ unit}}$

Ex-3) Find the radius of curvature of the curve $y = \log(\sin x)$ at $x = \frac{\pi}{2}$.

$\rightarrow$ Given equation of the curve is

$$y = \log(\sin x)$$

Differentiate w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dx}\log(\sin x) \qquad \left[\because \frac{d}{dx}\log(f(x)) = \frac{1}{f(x)}\cdot\frac{d}{dx}f(x)\right]$$

$$= \frac{1}{\sin x}\frac{d}{dx}(\sin x)$$

$$= \frac{\cos x}{\sin x} = \cot x$$

$$\therefore \frac{dy}{dx} = \cot x \quad \text{——— (1)}$$

$$\therefore \left[\frac{dy}{dx}\right]_{at\ x=\frac{\pi}{2}} = \cot\left(\frac{\pi}{2}\right) = \cot(90^\circ) = 0$$

Again differentiate (1) w.r.t. x

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(\cot x)$$

$$\therefore \frac{d^2y}{dx^2} = -\operatorname{cosec}^2 x$$

$$\therefore \left[\frac{d^2y}{dx^2}\right]_{at\ x=\frac{\pi}{2}} = -\operatorname{cosec}^2\left(\frac{\pi}{2}\right)$$

$$= -\operatorname{cosec}^2(90^\circ) = -(1)^2 = -1$$

$$\rho = \text{Radius of curvature} = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\left|\frac{d^2y}{dx^2}\right|}$$

$$= \frac{\left[1 + (0)^2\right]^{3/2}}{|-1|} = \frac{(1)^{3/2}}{1}$$

$$\therefore \boxed{\rho = 1 \text{ unit}}$$

Ex-4) Find the radius of curvature of the curve $xy = c$ at point (c, c).

$\longrightarrow$ Given equation of the curve

$$xy = c$$

Differentiate w.r.t. x

$$x\frac{dy}{dx} + y(1) = \frac{d}{dx}(c) \qquad \left[\because \frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}\right]$$

$$x\frac{dy}{dx} + y = 0$$

$$x\frac{dy}{dx} = -y$$

$$\therefore \frac{dy}{dx} = -\frac{y}{x} \quad \text{———} \quad (1)$$

$$\therefore \left[\frac{dy}{dx}\right]_{at\,(c,c)} = \frac{-c}{c} = -1$$

Again differentiate (1) w.r.t. x

$$\therefore \frac{d}{dx}\left(\frac{dy}{dx}\right) = -\frac{d}{dx}\left(\frac{y}{x}\right) \qquad \left[\because \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}\right]$$

Quotient Rule

$$\therefore \frac{d^2y}{dx^2} = -\left[\frac{x\frac{dy}{dx} - y\frac{d}{dx}(x)}{x^2}\right]$$

$$\frac{d^2y}{dx^2} = -\left[\frac{x\frac{dy}{dx} - y(1)}{x^2}\right]$$

$$\left[\frac{d^2y}{dx^2}\right]_{at\,(c,c)} = -\left[\frac{c(-1) - c}{c^2}\right] = -\left[\frac{-c-c}{c^2}\right] = -\left[\frac{-2c}{c^2}\right]$$

$$\left[\frac{d^2y}{dx^2}\right]_{at\,(c,c)} = \frac{2}{c}$$

$$\text{Radius of curvature} = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\left|\frac{d^2y}{dx^2}\right|}$$

$$\rho = \frac{[1 + (-1)^2]^{3/2}}{\left|\frac{2}{c}\right|} = \frac{(2)^{3/2}}{\left|\frac{2}{c}\right|}$$

$$\therefore \quad \rho = \frac{(2)^{3/2}}{\left(\frac{2}{c}\right)} = \frac{(2)^{3/2} \cdot c}{2}$$

$$\rho = \frac{(2.828) \cdot c}{2} = (1.414)c$$

$$\therefore \quad \boxed{\rho = (1.414) \cdot c \text{ unit}}$$

Ex-5) Find radius of curvature of the curve

$$\sqrt{x} + \sqrt{y} = 1 \quad \text{at} \quad \left(\frac{1}{4}, \frac{1}{4}\right)$$

→ Given equation of the curve

$$\sqrt{x} + \sqrt{y} = 1$$

Differentiate w.r.t. x

$$\therefore \frac{d}{dx}(\sqrt{x}) + \frac{d}{dx}(\sqrt{y}) = \frac{d}{dx}(1)$$

$$\therefore \frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}} \frac{dy}{dx} = 0 \qquad \left[\because \frac{d}{dx}\left(\sqrt{f(x)}\right) = \frac{1}{2\sqrt{f(x)}} \frac{d}{dx} f(x)\right]$$

$$\therefore \frac{1}{2\sqrt{y}} \frac{dy}{dx} = -\frac{1}{2\sqrt{x}}$$

$$\frac{dy}{dx} = \frac{-1}{2\sqrt{x}} \times \frac{2\sqrt{y}}{1}$$

$$\therefore \quad \frac{dy}{dx} = -\frac{\sqrt{y}}{\sqrt{x}} \quad \text{——} \quad (1)$$

$$\left[\frac{dy}{dx}\right]_{at\left(\frac{1}{4}, \frac{1}{4}\right)} = -\frac{\sqrt{\frac{1}{4}}}{\sqrt{\frac{1}{4}}} = -1$$

Again differentiate (1) w.r.t. x

$$\therefore \frac{d}{dx}\left(\frac{dy}{dx}\right) = -\frac{d}{dx}\left(\frac{\sqrt{y}}{\sqrt{x}}\right) \qquad \left[\because \text{By Quotient Rule} \quad \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}\right]$$

By Quotient Rule

$$\therefore \frac{d^2y}{dx^2} = -\left[\frac{\sqrt{x}\frac{d}{dx}(\sqrt{y}) - \sqrt{y}\frac{d}{dx}(\sqrt{x})}{(\sqrt{x})^2}\right]$$

$$= -\left[\frac{\sqrt{x}\frac{1}{2\sqrt{y}}\frac{dy}{dx} - \sqrt{y} \cdot \frac{1}{2\sqrt{x}}}{x}\right]$$

Subject :- Basic Mathematics (311302)

Question Bank

Unit - 4 :- $\boxed{\text{Differential Calculus}}$

2 Marks Questions

Q.1) If $f(x) = x^3 - 9x + 1$, find $f(2), f(3)$

Q.2) Find $\frac{dy}{dx}$ if $y = e^x \cdot \sin x$.

Q.3) Find $\frac{dy}{dx}$, if $y = x^{10} + 10^x + e^x + a^a$.

Q.4) Find slope of tangent to the curve $y = x^3$ at $x = 4$.

Q.5) If $f(x) = x^4 - 2x + 7$, then find $f(0) + f(2)$.

Q.6) If $f(x) = 3x^2 - 5x + 7$, show that $f(-1) = 3f(1)$.

Q.7) Find $\frac{dy}{dx}$, if $y = a^x + x^a + e^a + \log_a x$.

Q.8) Find $\frac{dy}{dx}$, if $y = e^x \cdot \sin^{-1} x$.

Q.9) If $f(x) = x^3 - \frac{1}{x^3}$, show that $f(x) + f\left(\frac{1}{x}\right) = 0$.

Q.10) State whether the function $f(x) = \frac{e^x + e^{-x}}{2}$ is even or odd.

Q.11) Find $\frac{dy}{dx}$ if $y = x^2 \cdot e^x$.

4 Marks Questions

Q.12) If $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$, then find $\frac{dy}{dx}$ at $\theta = \frac{\pi}{4}$.

Q.13) Find $\frac{dy}{dx}$, if $y = (\sin x)^x$.

Q.14) Find $\frac{dy}{dx}$, if $x^2 + y^2 = 4xy$.

Q.15) Find $\frac{dy}{dx}$, if $y = \tan^{-1}\left[\frac{a+x}{1-ax}\right]$

Q.16) A metal wire 36 cm long bent to form a rectangle, Find its dimensions when area is maximum.

Q.17) Find $\frac{dy}{dx}$ if $\sin y = \log(x+y)$

Q.18) If $x = a(2\theta - \sin 2\theta)$, $y = a(1-\cos 2\theta)$ find $\frac{dy}{dx}$ at $\theta = \frac{\pi}{4}$

Q.19) Find $\frac{dy}{dx}$, if $y = x^x + a^x + a^a$.

Q.20) Find $\frac{dy}{dx}$ at $\theta = \frac{\pi}{4}$, if $x = a\cos\theta$, $y = b\sin\theta$.

Q.21) If $x^y = e^{x-y}$, show that $\frac{dy}{dx} = \frac{\log x}{(1+\log x)^2}$

Q.22) If $y = \tan^{-1}\sqrt{\frac{1-\cos(2x)}{1+\cos(2x)}}$, then find $\frac{dy}{dx}$.

Q.23) Find $\frac{dy}{dx}$, if $x^3 + y^3 = 30xy$.

Q.24) If $x = a\cos^3\theta$ and $y = a\sin^3\theta$, find $\frac{dy}{dx}$ at $\theta = \frac{\pi}{3}$.

Q.25) Find maximum and minimum value of the function $y = 2x^3 - 21x^2 + 36x - 20$.

6 Marks Questions

Q.26) A bullet is fired into a mud tank and pentrates $(120t - 3600t^2)$ meters in t seconds after impact. Calculate maximum depth of penetration.

Q.27) Find the radius of curvature of the curve $y^2 = 4ax$ at point $(a, 2a)$.

Q.28) A metal wire 100 cm long is bent to form a rectangle. Find its dimension when its area is maximum.

Q.29) A telegraph wire hangs in the form of curve $y = a\log\left[\sec\left(\frac{x}{a}\right)\right]$, where 'a' is constant. Show that radius of curvature at any point is $a\sec\left(\frac{x}{a}\right)$.

Q.30) i) Find the radius of curvature of the curve $y = x^3$ at point $(2, 8)$.
ii) Find $\frac{dy}{dx}$, if $y = x^{\sin x}$.

Q.31) i) A beam is bent in the form of curve $y = 2\sin x - \sin(2x)$. Find the radius of curvature of beam at point $x = \frac{\pi}{2}$
ii) Find equation of tangent to the curve $4x^2 + 9y^2 = 40$ at $(1, 2)$.